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The Bochner-Radon-Nikodym theory for vector-valued measures

Mykola Ivanovich Yaremenko[1*]
[1]The National Technical University of Ukraine,
"Igor Sikorsky Kyiv Polytechnic Institute",
37, Prospect Beresteiskyi (former Peremohy), Kyiv, Ukraine, 03056,
ORCID iD: 0000-0002-9209-6059.

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Abstract: In this paper, we investigate the vector-valued measures, absolutely continuous measures, and the Bochner-Radon-Nikodym property for Banach spaces. These types of studies have a plethora of applications in stochastic processes, representation theory, and recently in neural nets. We establish the necessary and sufficient conditions that Banach space possesses the Radon-Nikodym property.

Key words: Banach space, conditional expectation, Radon-Nikodym property, Radon-Nikodym theorem, dentability, dentable set, Bochner integral.
Subject classification codes: 46B22, 46B25, 46G05, 46G10.

1. Introduction

This article is dedicated to the Phillips-Radon-Nikodym theory for Banach space vector measures. In 1933, the vector-valued measures were introduced by S. Bochner in [4], where the concept of integration was extended to vector-valued functions such generalization is implemented as the limit of integrals of simple functions, first, defining the integral of simple functions. Recently, vector-valued measures and Banach space integrals have found wide applications in neural networks, probability theory, and stochastic processes of martingale-type [11] and references therein.

Traditionally, the Radon-Nikodym theorems have three different intertwined features: measure theoretical, structural geometrical, and operator theoretic. In [4], S. Bochner constructed the integral of vector-valued functions and showed each vector-valued function with a bounded variation on the unit interval, which has almost everywhere derivative can be restored by the Bochner integral procedure almost everywhere. G. Birkhoff generalized Bochner's results to include infinitely dimensional Hilbert and some Banach spaces [2], G. Birkhoff showed that, in Hilbert spaces, an absolutely continuous function can be recovered from its derivative by Bochner integration procedure, in modern interpretation, a vector-valued function determines a continuous linear functional in the given Hilbert space via scalar product then application of the Riesz representation theorem provides the existence of the Riesz representation that provides the wanted derivative, namely, there is a connection between an absolutely continuous function and a uniquely defined vector via the inner product [2, 3].

A review of classical literature can be found in the works of J. Diestel J.J. Uhl [7], P. R. Halmos [10], and N. Bourbaki [3]; contemporary views in research of Z. Wang, G. J. Klir [22, 23, 24] and recent paper of S. Okada, J. Rodriguez, and E. A. Sanchez-Perez [19]; D. Chen [5]; M. Martin and A. Rueda Zoca [18];

*Correspondence: Math.kiev@gmail.com

and references therein. Structural features of the Radon-Nikodym theory were studied by M.A. Rieffel in [20, 21], who developed the notion of dentablity for general Banach spaces and showed that dentablity equals the possession of the Radon-Nikodym property for a given Banach space.

The remainder of the paper is organized as follows. In Section 2, we consider properties of Bochner integral in Banach spaces and some of their possible generalizations, In Section 3, we investigate the connection between the geometrical structure of a Banach space and its possession of the Radon-Nikodym property due to Rieffel ideals of dantablity of subsets of the Banach space. Section 4 is dedicated to three weak variants of the Radon-Nikodym theorem.

2. The Radon-Nikodym property for Banach spaces

Let X, Y and Z be separable reflexive Banach spaces. Let (E, Σ, μ) be a σ-finite measure space. Let be a -finite measure with finite variation .

We assume $\Lambda : X \times Y \to Z$ is a bilinear bounded mapping such that

$$\|\Lambda(a,b)\|_Z \le \|a\|_X \|b\|_Y \tag{1}$$

for all $a \in X$ and $b \in Y$. We denote $\Lambda(a,b) \equiv ab$ for all $a \in X, b \in Y$.

Definition 2.1. A σ-finite measure μ is called decomposable if there is a collection $\{D_\alpha\}$ of disjoint sets such that $\mu(D_\alpha) < \infty$ for all $\alpha$and $\mu(D) = \sum_\alpha \mu(D \cap D_\alpha)$ for all .

We always suppose that the measure μ is decomposable.

Definition 2.2. For each $D \in \Sigma$, the variation of a vector measure η is the scalar set function defined by

$$Var(\eta)(D) = |\eta|(D) = \sup_{\Pi(D)} \sum_{B \in \Pi(D)} \|\mu(B)\|_X, \tag{2}$$

where supremum is taken for all collections $\{B\}$ of partitions $\Pi(D)$ of a set D.

Definition 2.3. The general Bochner integral, of a simple function $\chi : E \to Y$ defined by

$$\chi(x) = \sum_{i=1,\dots,k} a_i 1_{B_i}(x)$$

here 1_{B_i} is the indicator functions of disjoint sets $B_i \in \Sigma$ and $a_i \in Y$, is defined by

$$\int_E \chi(x)\, d\eta(x) = \sum_{i=1,\dots,k} \Lambda(\eta(B_i), a_i) \equiv \sum_{i=1,\dots,k} a_i \eta(B_i) \in Z. \tag{3}$$

Definition 2.4. The space $S(E, Y, \mu)$ consisted of all simple functions $\chi : E \to Y$. A sequence $\{\chi_k\} \subset S(E, Y, \mu)$ is called to be fundamental in mean if

$$\lim_{k,m\to\infty} N(\chi_k - \chi_m) = \lim_{k,m\to\infty} \int_E \|\chi_k - \chi_m\|_Y \, d\mu = 0. \tag{4}$$

Definition 2.5. A function $f : E \to Y$ is vector integrable with respect to vector measure η if there exists a fundamental sequence $\{\chi_k\}$ of step functions convergent μ-almost everywhere to f and the integral of f is given by

$$\int_E f d\eta = \lim_{k\to\infty} \int_E \chi_k d\eta. \tag{5}$$

Definition 2.6. The space $L^1(E, Y, \eta)$ is a vector space of all η-integrable functions $f : E \to Y$ and with linear operation defined by

$$\int_E (\alpha f + \beta g)\, d\eta = \alpha \int_E f d\eta + \beta \int_E g d\eta \in Z \tag{6}$$

for all $f, g \in L^1(E, Y, \eta)$ and all scalars α and β.

Lemma 2.1. *Let $\nu : \Sigma \to Z$ be a σ-additive vector measure and $f : E \to Z$ be a function such that for each $\xi \in Z^*$ we have*

$$\langle \nu(D), \xi\rangle = \int_D \langle f, \xi\rangle\, d\mu \tag{7}$$

for all . Let $f : E \to Z$ be ν-measurable, then the identity

$$\nu(D) = \int_D f d\mu \tag{8}$$

holds for all .

Proof. Indeed, for each , there is an increasing sequence of sets $\{D_k\} \subset \Sigma$ such that $\mu(D_k) < \infty$ for all k and $D = \bigcup_k \{x \in D : \|f(x)\|_Z \le k\}$ so that every set is a non-more than a countable union of sets of D of finite measure μ. So, we have

$$\left\langle \int_{D_k} f d\mu, \xi \right\rangle = \int_{D_k} \langle f, \xi\rangle\, d\mu$$

for each $\xi \in Z^*$, and

$$\int_{D_k} \langle f, \xi\rangle\, d\mu = \langle \nu(D_k), \xi\rangle$$

and

$$\langle \nu(D_k), \xi\rangle = \left\langle \int_{D_k} f d\mu, \xi \right\rangle,$$

therefore, we conclude

$$\nu(D_k) = \int_{D_k} f d\mu$$

and for finite measure, the statement is proven. Now, let $\tilde{\xi} \in Z^*$ then a measure $\nu\left(\tilde{\xi}\right)$ be defined by

$$\nu\left(\tilde{\xi}\right)(C) = \left\langle \nu(C), \tilde{\xi}\right\rangle$$

for all $C \in \Sigma$. We obtain

$$\nu\left(\tilde{\xi}\right)(D_k) = \left\langle \nu(D_k), \tilde{\xi} \right\rangle = \left\langle \int_{D_k} f d\mu, \tilde{\xi} \right\rangle = \int_{D_k} \left\langle f, \tilde{\xi} \right\rangle d\mu.$$

Thus, we obtain the Pettis-Bochner integral formula

$$\nu(D) = \int_D f d\mu$$

for all . $\square$

Theorem 2.1. *Let $\eta : \Sigma \to LB(Y, Z)$ be a -finite measure with finite variation $\phi = Var(\eta)$. Then, the density $\Psi : E \to LB(Y, Z)$ is such that the equality*

$$\eta(D)\zeta = \int_D \Psi(x)\zeta d\mu(x) \tag{9}$$

holds for all $\zeta \in Y$ and all .

Proof. If a function $f : E \to Z$ is weakly μ- measurable and locally separable then the function f is strongly μ-measurable. Then, the function $f : E \to Z$ is defined by $f \equiv \Psi(x)\zeta$, which proves the theorem.

Now, we assume Π is the set of all finite partitions $\pi = \{B_i\}$ of $B \in \Sigma$ so that $\mu(B_i) > 0$. We assume that, on Π, there is the order of refinement of partitions. For each partition $\pi = \{B_i\}$, we introduce the step function

$$f_\pi = \sum_i \frac{\eta(B_i)\zeta}{\mu(B_i)}$$

so that

$$\int_{B_i} f_\pi d\mu = \eta(B_i)\zeta$$

for all $\zeta \in Y$ and all $B_i \in \Sigma$.

Each partition $\pi \in \Pi$ generates the σ-ring $R(\pi)$, which is the collection of all finite unions of sets of the partition π. Let $\{\pi_k\}$ is an increasing sequence of partitions $\pi_k \in \Pi$. Then, $\{R(\pi_k)\}$ there is an increasing sequence of σ-rings so that there is a union $\bigcup_k R(\pi_k)$ that is a countable ring. The ring $\bigcup_k R(\pi_k)$ generates the σ-ring $\Xi \subset \Sigma$. We consider restrictions $\tilde{\mu} = \mu|\Xi$ and $\tilde{\nu} = \eta(\cdot)\zeta|\Xi$. Then, $\tilde{\nu} : \Xi \to Z = LB(\bigcup_k R(\pi_k), Z)$ is a measure with finite variation $Var(\tilde{\nu}) \leq Var(\eta(\cdot)\zeta)$ for all $\zeta \in Y$ so that $\tilde{\nu} \ll \tilde{\mu}$. The average range $Ar(B)$ for all $B \in \Sigma$ of η is given by

$$Ar(B) = \left\{ \frac{\eta(D)}{\mu(D)} : D \in \Sigma \cap B \right\},$$

similarly, for all $\tilde{B} \in \Xi$, we have

$$Ar\left(\tilde{B}\right) = \left\{ \frac{\tilde{\nu}(D)}{\tilde{\mu}(D)} : D \in \Xi \cap \tilde{B} \right\}.$$

We denote $\tilde{\Psi}:\tilde{B}\rightarrow LB\left(\bigcup_k R\left(\pi_k\right),\tilde{Z}\right)$ so that

$$\tilde{\nu}\left(B\right)=\int_B \tilde{\Psi}d\tilde{\mu}$$

for all $B\in\Xi$.

For every k, we define a conditional expectation E_k by

$$E_k\left(f\right)=\sum_{D\in\pi_k}1_D\frac{1}{\tilde{\mu}\left(D\right)}\int_D fd\tilde{\mu},$$

we assume f is step function $f=\sum_i a_i 1_{B_i}\left(x\right)$ where $B_i\in\bigcup_k R\left(\pi_k\right)$ and $a_i\in Z$. For each step function f, we find a number $k\left(f\right)$ such that $B_i\in\bigcup_{k=1,\dots,k(f)}R\left(\pi_k\right)$ for all $k\geq k\left(f\right)$ so that

$$E_k\left(f\right)=\sum_{D\in\pi_k}1_D\frac{1}{\tilde{\mu}\left(D\right)}\sum_i a_i\tilde{\mu}\left(B\cap B_i\right)=f.$$

The operators E_k are contraction projections on $L^1\left(\tilde{\mu},Z\right)$ so that $E_k\left(f\right)\rightarrow f\in L^1\left(\tilde{\mu},Z\right)$ for all elements $f\in L^1\left(\tilde{\mu},Z\right)$ hence the set of all step functions is dense in $L^1\left(\tilde{\mu},Z\right)$, and,

$$\sum_{D\in\pi_k}1_D\frac{1}{\tilde{\mu}\left(D\right)}\sum_i a_i\tilde{\mu}\left(B_i\right)=f_{\pi_k}$$

so that the sequence $\left\{f_{\pi_k}\right\}$ is a fundamental in $L^1\left(\mu,\Sigma,Z\right)$.

We find a partition $\pi_D\in\Pi$ containing D so that

$$\int_D f_\pi d\mu=\int_D f_{\pi_D}d\mu=\eta\left(D\right)\zeta$$

for all $\pi\geq\pi_D$. Thus, we prove that

$$\eta\left(D\right)\zeta=\int_D fd\mu.$$

□

Statement 1. We show that the identity

$$Var\left(\eta\right)\left(D\right)=\int_D\left\|f\left(x\right)\right\|_Z d\mu\left(x\right)$$

holds for all $D\in\Sigma$.

Proof. Straightforwardly by N. Dinculeanu results, we have

$$Var\left(\eta\right)\left(D\right)\leq\int_D\left\|f\left(x\right)\right\|_Z d\mu\left(x\right)$$

for all $D \in \Sigma$. To show the reverse, we consider a fixed $D \in \Sigma$, $\varepsilon > 0$ and a disjoint partition $\pi = \{B_i\}$ so that $\bigcup_i B_i = D$, we obtain

$$\int_D |\langle f(x), \xi\rangle| \, d\mu(x) \le \sum_i |\langle \eta(B_i), \xi\rangle| + \varepsilon$$
$$\le \sum_i \|\eta(B_i)\|_Z + \varepsilon \le Var(\eta)(D) + \varepsilon,$$

where $\xi \in Z$ and

$$\|f(x)\|_Z = \sup_{\xi \in Z^*} \{|\langle f(x), \xi\rangle| : \|\xi\|_{Z^*} \le 1\}$$

for all $x \in E$. By the Radon-Nikodym theorem, there is a positive function ψ such that

$$Var(\eta)(D) = \int_D \psi(x) \, d\mu(x)$$

for all $D \in \Sigma$. Thus, we deduce

$$|\langle f(x), \xi\rangle| \le \psi(x)$$

for μ almost all x.

□

3. Deniability and its application to Radon-Nikodym property

In 1967, M.A. Rieffel introduced the notion of dentability, which provides the tools to establish a correlation between the structure of a Banach space and the Radon-Nikodym property [20, 21].

Definition 3.1. Let Y be a Banach space. A bounded subset D of Y is called dentable if for any $\varepsilon > 0$ there is an element $\xi(\varepsilon) \in X$ such that $\xi(\varepsilon) \notin clos(hull(D \backslash B_\varepsilon(\xi(\varepsilon))))$, where $clos$ is the closure of a convex hull, and $B_\varepsilon(\xi(\varepsilon))$ is a ball of radius ε centered at $\xi(\varepsilon)$.

Statement (Rieffel) 2. All relative norm compact subsets of a Banach space Y are dentable.

Proof. For each extreme element ξ of norm compact convex subset D, we obtain that the closure of $D \backslash B_\varepsilon(\xi)$ does not contain ξ for any $\varepsilon > 0$. The Krein–Milman theorem yields that $\xi \notin clos(hull(D \backslash B_\varepsilon(\xi)))$.

Statement (Rieffel) 3. If $clos(hull(D))$ is dentable then subset D is also dentable.

Proof. Let $\varepsilon > 0$ then we find an element $\tilde{\xi} \notin clos(hull(D))$ such that

$$\tilde{\xi} \notin clos\left(hull\left(clos(hull(D)) \backslash B_{\frac{\varepsilon}{2}}\left(\tilde{\xi}\right)\right)\right).$$

We select

$$\xi \in D \backslash clos\left(hull\left(clos(hull(D)) \backslash B_{\frac{\varepsilon}{2}}\left(\tilde{\xi}\right)\right)\right)$$

then $\xi \in B_{\frac{\varepsilon}{2}}\left(\tilde{\xi}\right)$ therefore we have

$$\xi \notin clos(hull(D \backslash B_\varepsilon(\xi))).$$

□

Definition 3.2. Let Y be a Banach space. A bounded subset $D \in \Sigma$ is called (ξ, ε)-pure for vector measure η with respect to μ if $\frac{\eta(B)}{\mu(B)} \in B_\varepsilon(\xi)$ for all $B \subseteq D$ such that $\mu(B) < \infty$.

A convex subset $D \in Y$ is called strongly smoothable if there exists some $\xi \in Y \backslash clos(D)$ and some $\zeta \in S(Y^*) = \{\varsigma \in Y^* : \|\varsigma\| = 1\}$ such that

$$\left\{\tilde{\xi} \in B_1(Y) : \zeta\left(\tilde{\xi}\right) \geq \varepsilon\right\} \subset clos\left(\bigcup_\tau \{\tau(D - \xi) : \tau \geq 0\}\right)$$

for each $\varepsilon > 0$.

Let $X = \Theta^*$, then a convex subset $D \in Y$ is called weak-star smoothable if there exists some $\xi \in Y \backslash weak^* clos(D)$ and some $\zeta \in \Theta$ such that

$$\left\{\tilde{\xi} \in B_1(X) : \zeta\left(\tilde{\xi}\right) \geq \varepsilon\right\} \subset clos\left(\bigcup_\tau \{\tau(D - \xi) : \tau \geq 0\}\right)$$

for each $\varepsilon > 0$.

Proposition 3.1. *Let $D \in Y$ be a closed convex bounded subset and let $0 \in D$. Then, we have*

1) for a subset D to be dentable it is necessary and sufficient that the set $D^0 = \{\varsigma \in Y^* : \sup\langle\varsigma, D\rangle \leq 1\}$ be strongly smoothable;

2) for a subset D to be strongly smoothable it is necessary and sufficient that $D_0 = \left\{\tilde{\xi} \in \Theta : \sup\left\langle\tilde{\xi}, D\right\rangle \leq 1\right\}$ be weak-star dentable.

Proposition 3.2. *Let $D \in Y$ be a closed convex subset. Let $\tilde{C} \in Y^*$ be a closed convex weak-star subset. Then, we have*

1) D is dentable if and only if $weak^* clos(imag_{Y^{**}}(D))$ is weak-star dentable;

2) set $\tilde{C}$ is dentable if $\tilde{C}$ is weak-star dentable;

3) $weak^* clos(imag_{Y^{**}}(D))$ is weak-star strongly smoothable if D is strongly smoothable.

The main theorem for dentable spaces.

Theorem 3.1. *Let Y be a Banach space. Let (E, Σ, μ) be a -finite measure space. Let $\eta : \Sigma \to Y$ be a vector measure. Then, there exists a Bochner integrable function $f : E \to Y$ such that the identity*

$$\eta(D) = \int_D f(x)\, d\mu(x)$$

holds for all if and only if:

1) from $\mu(B) = 0$ follows $\eta(B) = 0$, $B \in \Sigma$;

2) $\phi = Var(\eta) < \infty$;

3) the average range of η is locally dentable set, namely, for given with finite measure, and any $\varepsilon > 0$ there exists a subset $B \subseteq D$ such that $\mu(D \backslash B) < \varepsilon$ and the average range.

$$Ar(B) = \left\{\frac{\eta\left(\tilde{B}\right)}{\mu\left(\tilde{B}\right)} : \tilde{B} \subseteq B, \mu\left(\tilde{B}\right) > 0\right\}$$

is a dentable set.

Proof. Let Π be the set of all finite disjoint partitions $\pi = \{B_i\}$ of $B \in \Sigma$ so that $\mu(B_i) > 0$ as previously. Then, Π is a directed set, for each partition $\pi \in \Pi$ we define a simple function by

$$f_\pi = \sum_{B\in\pi} \frac{\eta(B)\,1_B}{\mu(B)}.$$

For any $\varepsilon > 0$, we find a $\tilde{\pi} \in \Pi$ such that

$$\|f_\pi - f_{\tilde{\pi}}\| < \varepsilon$$

for all $\pi \succ \tilde{\pi}$. Since $\phi = Var(\eta) < \infty$ there is a finite measure set $C \in \Sigma$ such that $\phi(E\backslash C) < \frac{\varepsilon}{3}$. From 1), we can find $\delta > 0$ such that $\mu(B) < \delta$ follows $\phi(B) < \frac{\varepsilon}{6}$. □

Lemma 3.1. *(Rieffel) 2. Let Y be a Banach space. Let (E, Σ, μ) be a -finite measure space. Let $\eta : \Sigma \to Y$ be a vector measure. Let 1), 2), and 3) as above. Then, for any $\varepsilon > 0$, there are sequences $\{\xi_i\} \subset Y$ and $\{B_i\} \subset \Sigma$ such that B_i is (ξ, ε)-pure for vector measure η with respect to μ for all i, and $E = \bigcup_i B_i$.*

The proof is given by M.A. Rieffel in [20, 21].

By the Rieffel lemma, for any $\varepsilon > 0$, there are sequences $\{\xi_i\} \subset Y$ and $\{B_i\} \subset \Sigma$ such that disjointed B_i is $\left(\xi_i, \frac{\varepsilon}{6\mu(D)}\right)$-pure for vector measure η for all i, so that $\mu(B_i) > 0$ and $D = \bigcup_i B_i$. There is a number k such that $\mu\left(D\backslash \bigcup_{i=1,\ldots,k} B_i\right) < \delta$. If $\tilde{\pi} = \{B_i : i = 1, ..., k\}$ then we obtain $\|f_\pi - f_{\tilde{\pi}}\| < \varepsilon$ for all $\pi \succ \tilde{\pi}$. Since $\{f_\pi\}$ is a mean fundamental, we can find an integrable convergent in mean function f such that

$$\int_D f d\mu = \lim_\pi \int_D f_\pi d\mu$$

for all .

If $\mu(D) = 0$ then we have

$$\eta(D) = \int_D f d\mu$$

since the measure η is μ-continuous.

If $0 < \mu(D) < \infty$ then we obtain

$$\int_D f_\pi d\mu = \eta(D)$$

for all partitions $\pi \succ \tilde{\pi}$ thus we conclude

$$\eta(D) = \lim_\pi \int_D f_\pi d\mu = \int_D f d\mu,$$

which proves the theorem.

As a consequence, we have the following theorem.

Theorem 3.2. *The Banach space possesses the Radon-Nikodym property if and only if each closed convex bounded subset of this Banach space is dentable.*

Proof. We assume that Banach space Y possesses the Radon-Nikodym property. Then, we must show that each closed convex bounded subset D of Y is dentable. We assume the reverse that D is closed convex bounded non-dentable subset of Y and we presuppose that D belongs to unit ball in Y.

We take $0 < \varepsilon < 1$ such that from $\xi \in D$ follows $\xi \in clos\left(hull\left(D\backslash B_{\varepsilon}\left(\xi\right)\right)\right)$. We consider the interval $[0,1)$ with the standard Lebesgue measure λ on it. By induction, we define the monotone sequence $\{\Xi_k : \Xi_k \subset \Xi_{k+1}\}$ of finite set algebras and additive mappings

$$\{\vartheta_k : \Xi_k \to Y\}$$

such that

1) the atoms of Ξ_k partition $[0,1)$ into half-open intervals $\left\{I_{1,k}, I_{2,k}, ..., I_{j(k),k}\right\}$;

2) $\frac{\vartheta_k(I_{i,k})}{\lambda(I_{i,k})} \in D$ for all k and all $1 \le i \le j\left(k\right)$;

3) from $I_{l,k+1} \subset I_{i,k}$ follows

$$\left\|\frac{\vartheta_{k+1}\left(I_{i,k+1}\right)}{\lambda\left(I_{l,k+1}\right)} - \frac{\vartheta_{k+1}\left(I_{i,k}\right)}{\lambda\left(I_{i,k}\right)}\right\|_Y \ge \frac{2^k - 1}{2^k}\varepsilon;$$

4) we have $\left\|\vartheta_k\left(B\right) - \vartheta_{k+1}\left(B\right)\right\|_Y \le \frac{\varepsilon}{2^k}\lambda\left(B\right)$ for all $B \in \Xi_k$;

5) $\left\|\vartheta_k\left(B\right)\right\|_Y \le \lambda\left(B\right)$ for all $B \in \Xi_k$;

6)

$$\left\|\frac{\vartheta_k\left(B\right)}{\lambda\left(B\right)} - \frac{\vartheta_{k+1}\left(B\right)}{\lambda\left(B\right)}\right\|_Y \le \frac{\varepsilon}{2^k}$$

for all $B \in \Xi_k$ such that $\lambda\left(B\right) > 0$.

We take $\Xi_0 = \{\emptyset, [0,1)\}$ and any $\xi_0 \in D$ and we define $\vartheta_0\left(\emptyset\right) = 0$ and $\vartheta_0\left([0,1)\right) = \xi_0$. Assuming that D is non-dentable set then there is a set $\left\{\alpha_{1,1}, ..., \alpha_{j(1),1} : \alpha_{j(1),i} > 0\right\}$ such that $\sum_{i=1,...,j(1)} \alpha_{i,1} = 1$ and $\alpha_{i,1} \in D$. We have

$$\left\|\frac{\vartheta_0\left([0,1)\right)}{\lambda\left([0,1)\right)} - \alpha_{i,1}\right\|_Y \ge \left\|\xi_0 - \xi_{i,1}\right\|_Y \ge \varepsilon$$

and

$$\left\|\frac{\vartheta_0\left([0,1)\right)}{\lambda\left([0,1)\right)} - \sum_{i=1,...,j(1)} \alpha_{i,1}\xi_{i,1}\right\|_Y = \left\|\xi_0 - \sum_{i=1,...,j(1)} \alpha_{i,1}\xi_{i,1}\right\|_Y < \frac{\varepsilon}{2^k}.$$

We split $[0,1)$ into finite disjoint partition of half-open intervals $\left\{I_{1,1}, I_{2,1}, ..., I_{j(k),1}\right\}$ so that $\lambda\left(I_{i,1}\right) = \xi_{i,1}$. Then, the algebra Ξ_1 is a collection of subsets $[0,1)$ generated by $\left\{I_{1,1}, I_{2,1}, ..., I_{j(k),1}\right\}$. We define $\vartheta_1 : \Xi_1 \to Y$ by $\vartheta_1\left(I_{i,1}\right) = \alpha_{i,1}\xi_{i,1}$ for all $1 \le i \le j\left(1\right)$.

By induction, we assume that Ξ_k and ϑ_k are defined and Ξ_k is a disjoint collection of half-open intervals $\left\{I_{1,k}, I_{2,k}, ..., I_{j(k),k}\right\}$, and $\frac{\vartheta_k(I_{i,k})}{\lambda(I_{l,k})} \in D$ for each $1 \le i \le j\left(k\right)$, next, we define Ξ_{k+1} and ϑ_{k+1} so that there are $\left\{\alpha_1\left(l\right), ..., \alpha_{s(l)}\left(l\right) : \alpha_i\left(l\right) > 0\right\}$, $\sum_{i=1,...,s(l)} \alpha_i\left(l\right) = 1$, and $\left\{\xi_1\left(l\right), ..., \xi_{s(l)}\left(l\right) : \xi_i\left(l\right) \in D\right\}$ such that

$$\left\|\xi_i\left(l\right) - \frac{\vartheta_k\left(I_{l,k}\right)}{\lambda\left(I_{l,k}\right)}\right\|_Y \ge \varepsilon$$

and

$$\left\|\frac{\vartheta_k\left(I_{l,k}\right)}{\lambda\left(I_{l,k}\right)}-\sum_{i=1,\ldots,s(l)}\alpha_i\left(l\right)\xi_i\left(l\right)\right\|_Y<\frac{\varepsilon}{2^k}.$$

We split $I_{l,k}$ into pairwise disjoint half-open intervals $\left\{J_1\left(l\right),J_2\left(l\right),...,J_{s(l)}\left(l\right)\right\}$ such that $\lambda\left(J_i\left(l\right)\right)=\lambda\left(I_{l,k}\right)\alpha_i\left(l\right)$ for all $1\leq i\leq s\left(l\right)$. So, algebra Ξ_{k+1} is generated by $\left\{J_1\left(l\right),J_2\left(l\right),...,J_{s(l)}\left(l\right)\right\}$ for $1\leq l\leq j\left(k\right)$ and ϑ_{k+1} is defined $\vartheta_{k+1}\left(J_i\left(l\right)\right)=\lambda\left(J_i\left(l\right)\right)\xi_i\left(l\right)$ on Ξ_{k+1}.

It is easy to see that 1) -2) are satisfied. 3) is following from the estimate

$$\left\|\frac{\vartheta_{k+1}\left(J_i\left(l\right)\right)}{\lambda\left(J_i\left(l\right)\right)}-\frac{\vartheta_{k+1}\left(I_{l,k}\right)}{\lambda\left(I_{l,k}\right)}\right\|_Y=\geq\left\|\xi_i\left(l\right)-\sum_{i=1,\ldots,s(l)}\frac{\vartheta_{k+1}\left(J_i\left(l\right)\right)}{\lambda\left(I_{l,k}\right)}\right\|_Y$$

$$=\left\|\xi_i\left(l\right)-\sum_{i=1,\ldots,s(l)}\alpha_i\left(l\right)\xi_i\left(l\right)\right\|_Y\geq\left\|\xi_i\left(l\right)-\frac{\vartheta_k\left(I_{l,k}\right)}{\lambda\left(I_{l,k}\right)}\right\|_Y$$

$$-\left\|\sum_{i=1,\ldots,s(l)}\alpha_i\left(l\right)\xi_i\left(l\right)-\frac{\vartheta_k\left(I_{l,k}\right)}{\lambda\left(I_{l,k}\right)}\right\|_Y\geq\varepsilon-\frac{\varepsilon}{2^k}=\frac{2^k-1}{2^k}\varepsilon.$$

4) can be obtained as follows

$$\left\|\vartheta_k\left(I_{l,k}\right)-\vartheta_{k+1}\left(I_{l,k}\right)\right\|_Y=\left\|\vartheta_k\left(I_{l,k}\right)-\sum_{i=1,\ldots,s(l)}\vartheta_{k+1}\left(J_i\left(l\right)\right)\right\|_Y$$

$$=\left\|\vartheta_k\left(I_{l,k}\right)-\sum_{i=1,\ldots,s(l)}\lambda\left(I_{l,k}\right)\alpha_i\left(l\right)\xi_i\left(l\right)\right\|_Y$$

$$=\lambda\left(I_{l,k}\right)\left\|\frac{\vartheta_k\left(I_{l,k}\right)}{\lambda\left(I_{l,k}\right)}-\sum_{i=1,\ldots,s(l)}\alpha_i\left(l\right)\xi_i\left(l\right)\right\|_Y\leq\lambda\left(I_{l,k}\right)\frac{\varepsilon}{2^k}$$

so that

$$\left\|\vartheta_k\left(B\right)-\vartheta_{k+1}\left(B\right)\right\|_Y\leq\lambda\left(B\right)\frac{\varepsilon}{2^k}$$

for all $B\in\Xi_k$.

We denote $\Lambda=\bigcup_k\Xi_k$ an algebra of subsets of $[0,1)$. There is a limit

$$\lim_{k\to\infty}\vartheta_k\left(B\right)=\vartheta\left(B\right)$$

for all $B\in\Lambda$. The set function $\vartheta:\Lambda\to Y$ is σ-additive and strongly additive on $\Lambda=\bigcup_k\Xi_k$ and $\left\|\vartheta\left(B\right)\right\|_Y\leq\lambda\left(B\right)$ for all $B\in\Lambda$. Thus, the application of the Kluvanek extension theorem yields the existence of a σ -additive set function $\tilde{\vartheta}:\Xi\to Y$, which coincides with ϑ on Λ, where Ξ is a σ-algebra generated by set algebra Λ.

We have

$$\left|\left\langle f,\tilde{\vartheta}(B)\right\rangle\right| = \left|\lim_{\pi}\sum\left\langle f,\tilde{\vartheta}(B_i)\right\rangle\right|$$
$$= \lim_{\pi}\left|\sum\left\langle f,\tilde{\vartheta}(B_i)\right\rangle\right| \leq \lim_{\pi}\sum\left|\left\langle f,\tilde{\vartheta}(B_i)\right\rangle\right|$$
$$\leq \lim_{\pi}\sum\lambda(B_i) = \lambda(B)$$

so that $\left\|\tilde{\vartheta}(B)\right\|_Y \leq \lambda(B)$ for all $B \in \Xi$, which means that $\tilde{\vartheta}$ has a bounded variation.

By the Radon-Nikodym property, we have that function $\tilde{\vartheta}$ is differentiable with respect to measure λ so that

$$\tilde{\vartheta}(B) = \int_B f d\lambda$$

for all $B \in \Xi$.

For fixed $\tilde{B} \in \Xi$, $\lambda\left(\tilde{B}\right) > 0$, we can find $B \in \Xi$, $B \subset \tilde{B}$ and $\lambda(B) > 0$ such that $0 < Ar\left(\tilde{B}\right) < \frac{\varepsilon}{10}$, but if we prove that $Ar\left(\tilde{B}\right) \geq \frac{\varepsilon}{4}$ for all $B \in \Xi$ such that $\lambda(B) > 0$ then we obtain the contradiction and each subset must be dentable.

For $4 \leq k$, we have

$$\left\|\frac{\tilde{\vartheta}_{k+1}(I_{l,k+1})}{\lambda(I_{l,k+1})} - \frac{\tilde{\vartheta}(I_{l,k})}{\lambda(I_{l,k})}\right\|_Y \geq \frac{\varepsilon}{2}$$

and

$$\left\|\tfrac{\vartheta_t(I_{l,k+1})}{\lambda(I_{l,k+1})} - \tfrac{\vartheta_t(I_{l,k})}{\lambda(I_{l,k})}\right\|_Y \geq \left\|\tfrac{\vartheta_{k+1}(I_{l,k+1})}{\lambda(I_{l,k+1})} - \tfrac{\vartheta_{k+1}(I_{l,k})}{\lambda(I_{l,k})}\right\|_Y - \left\|\tfrac{\vartheta_k(I_{l,k})}{\lambda(I_{l,k})} - \tfrac{\vartheta_{k+1}(I_{l,k})}{\lambda(I_{l,k})}\right\|_Y$$
$$- \left\|\tfrac{\vartheta_{k+1}(I_{l,k+1})}{\lambda(I_{l,k+1})} - \tfrac{\vartheta_t(I_{l,k+1})}{\lambda(I_{l,k+1})}\right\|_Y$$
$$\geq \tfrac{2^k-1}{2^k}\varepsilon - \tfrac{\varepsilon}{2^k} - \textstyle\sum_{q=k+1,\ldots,t-1}\left\|\tfrac{\vartheta_q(I_{l,k+1})}{\lambda(I_{l,k+1})} - \tfrac{\vartheta_{k+1}(I_{l,k+1})}{\lambda(I_{l,k+1})}\right\|_Y$$
$$\geq \tfrac{2^k-1}{2^k}\varepsilon - \tfrac{\varepsilon}{2^k} - \textstyle\sum_{q=k+1,\ldots,t-1}\tfrac{\varepsilon}{2^q} > \tfrac{\varepsilon}{2},$$

where $t \geq k+1$. We pass to the limit as $t \to \infty$.

For each $C \in \Xi$, $\lambda(C) > 0$, there is a subset $B \in \Lambda$ such that

$$\lambda(B\backslash C) + \lambda(B\backslash C) < \frac{\varepsilon}{16}\lambda(C)$$

so that

$$\lambda(B\backslash C) < \tfrac{\varepsilon}{16}\lambda(C) - \lambda(C\backslash B) \leq$$
$$\leq \tfrac{\varepsilon}{16}\lambda(C \cap B).$$

Since B is in Ξ_k for some $k \geq 4$ we have $B = \bigcup_{i\in D} I_{i,k}$. For some $i_0 \in D$, we have

$$0 < \lambda(I_{i_0,k}\backslash C) < \frac{\varepsilon}{16}\lambda(I_{i_0,k} \cap C)$$

and

$$0 < \lambda(B\backslash C) < \frac{\varepsilon}{16}\lambda(B \cap C)$$

so that $0<\lambda\left(I_{i_0,k}\backslash\left(I_{i_0,k}\cap C\right)\right)<\frac{\varepsilon}{16}\lambda\left(I_{i_0,k}\cap C\right)$. Thus, we have $I_{i_1,k}\subset I_{i_0,k}$ so that

$$0<\lambda\left(I_{i_1,k+1}\backslash\left(I_{i_1,k}\cap\left(I_{i_0,k}\cap C\right)\right)\right)<\frac{\varepsilon}{16}\lambda\left(\left(I_{i_1,k}\cap\left(I_{i_0,k}\cap C\right)\right)\right).$$

We obtain

$$\left\|\frac{\tilde{\vartheta}\left(I_{i_1,k+1}\right)}{\lambda\left(I_{i_1,k+1}\right)}-\frac{\tilde{\vartheta}\left(I_{i_0,k}\right)}{\lambda\left(I_{i_0,k}\right)}\right\|_Y\geq\frac{\varepsilon}{2},$$

$$\left\|\frac{\tilde{\vartheta}\left(I_{i_0,k}\right)}{\lambda\left(I_{i_0,k}\right)}-\frac{\tilde{\vartheta}\left(I_{i_0,k}\cap C\right)}{\lambda\left(I_{i_0,k}\cap C\right)}\right\|_Y=\frac{\lambda\left(I_{i_0,k}\backslash\left(I_{i_0,k}\cap C\right)\right)}{\lambda\left(I_{i_0,k}\cap C\right)}\left\|\frac{\tilde{\vartheta}\left(I_{i_0,k}\backslash\left(I_{i_0,k}\cap C\right)\right)}{\lambda\left(I_{i_0,k}\backslash\left(I_{i_0,k}\cap C\right)\right)}-\frac{\tilde{\vartheta}\left(I_{i_0,k}\right)}{\lambda\left(I_{i_0,k}\right)}\right\|_Y$$

and

$$\left\|\frac{\tilde{\vartheta}\left(I_{i_1,k+1}\right)}{\lambda\left(I_{i_1,k+1}\right)}-\frac{\tilde{\vartheta}\left(I_{i_1,k}\cap\left(I_{i_0,k}\cap C\right)\right)}{\lambda\left(I_{i_1,k}\cap\left(I_{i_0,k}\cap C\right)\right)}\right\|_Y\leq\frac{\varepsilon}{8}.$$

Thus, we conclude

$$\left\|\frac{\tilde{\vartheta}\left(I_{i_0,k}\cap C\right)}{\lambda\left(I_{i_0,k}\cap C\right)}-\frac{\tilde{\vartheta}\left(I_{i_1,k}\cap\left(I_{i_0,k}\cap C\right)\right)}{\lambda\left(I_{i_1,k}\cap\left(I_{i_0,k}\cap C\right)\right)}\right\|_Y\geq\frac{\varepsilon}{4},$$

thus, we obtain $Ar\left(\vartheta\right)\geq\frac{\varepsilon}{4}$, which proves the theorem. □

Examples. All reflective spaces have the Radon-Nikodym property. Lebesgue space $L^1\left([0,1]\right)$ and space C_0 do not possess the Radon-Nikodym property.

4. The Radon-Nikodym theory for general Bochner's integral constructions

We assume that $\eta:\Sigma\to LB\left(Y,Z\right)$ is a σ-finite measure with finite variation $\phi=Var\left(\eta\right)$, and $\Lambda:X\times Y\to Z$ is a bilinear bounded mapping, which will be denoted by $\Lambda\left(a,b\right)=ab$ for all $a\in X,b\in Y$.

Theorem 4.1. *Let ϕ be a scalar σ-finite measure and let the measure η be absolutely continuous with respect to ϕ. Then, there exists a function*

$$\Psi\left(\eta\right)\left(\cdot\right):E\to LB\left(Y,Z\right)\tag{10}$$

such that

1) we have

$$\left\|\Psi\left(\eta\right)\left(x\right)\right\|_{Y\to Z}\overset{\phi-a.e.}{=}1;$$

2) let $\langle\cdot,\cdot\rangle:Z\times Z^*\to R$ be duality pairing then $\langle\Psi\left(\eta\right)\left(f\right),\zeta\rangle\in L^1\left(\phi\right)$ for all $\zeta\in Z^*$ and the identity

$$\left\langle\int_D f\left(x\right)d\eta\left(x\right),\zeta\right\rangle=\int_D\left\langle\Psi\left(\eta\right)\left(x\right)f\left(x\right),\zeta\right\rangle d\phi\left(x\right)\tag{11}$$

holds for all $f\in L^1\left(E,Y,\eta\right)$ and all $\zeta\in Z^*$ and all $D\in\Sigma$.

Proof. If $\eta = 0$ then $\Psi(\eta)(\cdot) = 0$, the theorem has been proven. We presume $\eta \neq 0$. We fix $\xi \in Y$ and $\zeta \in Z^*$, and define the scalar measure by

$$\eta(\xi,\zeta)(D) = \langle \eta(D)\xi, \zeta \rangle$$

for all $D \in \Sigma$, so that

$$|\eta(\xi,\zeta)| \leq \phi \|\xi\|_Y \|\zeta\|_{Z^*}.$$

The classical Radon-Nikodym theorem yields the existence of a scalar locally ϕ-integrable function $\psi_{\xi\zeta}$ so that identities

$$\eta(\xi,\zeta) = \phi\psi_{\xi\zeta}$$

and

$$|\eta(\xi,\zeta)| = \phi|\psi_{\xi\zeta}|$$

hold ϕ-almost everywhere. For each $x \in E$ and $\xi \in Y$, a continuous linear functional on Z^* is given by

$$\psi_\xi(x) : \xi \to \psi_{\xi\zeta}(x)$$

and such that

$$|\psi_\xi(x)| \leq \|\xi\|_Y.$$

For each fixed $x \in E$, a linear continuous mapping is given by

$$\Psi(\eta)(x)(\cdot) : \xi \to \psi_\xi(x)$$

so that

$$\|\Psi(\eta)(x)\|_{Y \to Z} \leq 1.$$

For all $x \in E$, $\xi \in Y$, $\zeta \in Z^*$, we conclude

$$\langle \Psi(\eta)(x)\xi, \zeta \rangle = \psi_{\xi\zeta}(x)$$

so that

$$\langle \eta(D)\xi, \zeta \rangle = \int_D \langle \Psi(\eta)(x)\xi, \zeta \rangle \, d\phi(x)$$

and

$$\left\langle \int_D \chi d\eta, \zeta \right\rangle = \int_D \langle \Psi(\eta)\chi(x), \zeta \rangle \, d\phi(x)$$

for all $D \in \Sigma$, where $\chi : E \to Y$ is an arbitrary simple function.

Let $f \in L^1(E, Y, \eta)$ then there exists a sequence $\{\chi_k\}$ of simple functions that converges in mean and ϕ-almost everywhere to the function f. There is a number k_0 such that the inequality

$$|\langle \Psi(\eta)\chi_k, \zeta \rangle - \langle \Psi(\eta)\chi_m, \zeta \rangle| \leq \|\chi_k - \chi_m\|_Y \|\zeta\|_{Z^*}$$

holds for all $k, m \geq k_0$. Therefore, a sequence $\{\langle \Psi(\eta)\chi_k, \zeta \rangle\}$ is fundamental in and a sequence $\{\langle \Psi(\eta)\chi_k, \zeta \rangle\}$ converges ϕ-almost everywhere to $\langle \Psi(\eta)f, \zeta \rangle$. Thus, for all $\zeta \in Z^*$, we obtain

$$\lim_{k\to\infty} \int_D \langle \Psi(\eta)\chi_k, \zeta \rangle \, d\phi = \int_D \langle \Psi(\eta)f, \zeta \rangle \, d\phi$$

$$\lim_{k\to\infty}\left\langle \int_D \chi_k d\eta, \zeta \right\rangle = \left\langle \int_D f d\eta, \zeta \right\rangle,$$

thus

$$\lim_{k\to\infty}\int_D \chi_k d\eta = \int_D f d\eta.$$

Hence

$$\left\langle \int_D \chi_k d\eta, \zeta \right\rangle = \int_D \left\langle \Psi\left(\eta\right)\chi_k, \zeta\right\rangle d\eta$$

we have

$$\left\langle \int_D f d\eta, \zeta \right\rangle = \int_D \left\langle \Psi\left(\eta\right) f, \zeta\right\rangle d\eta$$

for all $\zeta \in Z^*$.

We estimate

$$\left|\left\langle \eta\left(D\right)\xi, \zeta\right\rangle\right| \leq \int_D \phi \left\|\xi\right\|_Y \left\|\zeta\right\|_{Z^*} \left\|\Psi\left(\eta\right)\left(x\right)\right\|_{Y\to Z} d\phi\left(x\right).$$

We denote $\varpi = \left\|\Psi\left(\eta\right)\right\|_{Y\to Z}\phi$ and we calculate

$$\left\|\eta\left(D\right)\right\|_{Y\to Z} \leq \int_D \left\|\Psi\left(\eta\right)\left(x\right)\right\|_{Y\to Z} d\phi\left(x\right) = \varpi\left(D\right).$$

The variation ϕ is the least positive measure for which the inequality $\left\|\eta\left(D\right)\right\|_{Y\to Z} \leq \phi\left(D\right)$ holds for all $D \in \Sigma$. Thus, we deduce $\left\|\Psi\left(\eta\right)\left(x\right)\right\|_{Y\to Z} \overset{\phi-a.e.}{=} 1$. The theorem has been proven. □

Theorem 4.2. *Let μ be a σ-finite scalar measure on Σ. Let $\Upsilon : E \to LB\left(Y,Z\right)$ be a function such that $\left\|\Upsilon\right\|_{Y\to Z}$ is locally μ-integrable, and let $\left\langle \Upsilon\xi, \zeta\right\rangle$ is μ-integrable for all $\xi \in Y$ and all $\zeta \in Z^*$. Then, there exists a measure $\eta : \Sigma \to LB\left(Y,Z\right)$ with ϕ such that*

$$\left\langle \int_D f\left(x\right) d\eta\left(x\right), \zeta \right\rangle = \int_D \left\langle \Upsilon\left(x\right) f\left(x\right), \zeta\right\rangle d\mu\left(x\right) \tag{12}$$

for all $f \in L^1\left(E,Y,\phi\right)$ and all $\zeta \in Z^*$ and all $D \in \Sigma$; and

$$\int_D \left|\varphi\right|\left(x\right) d\phi\left(x\right) = \int_D \left\|\Upsilon\left(x\right)\right\|_{Y\to Z} \left|\varphi\right|\left(x\right) dVar\left(\mu\right)\left(x\right) \tag{13}$$

for all $\varphi \in L^1\left(\phi\right)$ and all $D \in \Sigma$, where $\phi = \left\|\Upsilon\right\|_{Y\to Z} Var\left(\mu\right)$.

Proof. We define a set function

$$\eta\left(\xi,\zeta\right)\left(D\right) = \int_D \left\langle \Upsilon\xi, \zeta\right\rangle d\mu$$

for all $\xi \in Y$, $\zeta \in Z^*$, and $D \in \Sigma$. The continuous linear functional

$$\eta\left(\xi\right)\left(D\right) : \zeta \mapsto \eta\left(\xi,\zeta\right)\left(D\right)$$

satisfies the inequality

$$\|\eta(\xi)(D)\|_Z \leq \|\xi\|_Y \int_D \|\Upsilon\|_{Y\to Z}\, dVar(\mu).$$

We have

$$\langle \eta(D)\xi, \zeta\rangle = \int_D \langle \Upsilon\xi, \zeta\rangle\, d\mu$$

for all $\xi \in Y$, $\zeta \in Z^*$, and $D \in \Sigma$.

Let $\chi : E \to Y$ be a simple function given by

$$\chi(x) = \sum_{i=1,\dots,k} a_i 1_{B_i}(x),$$

where 1_{B_i} is the indicator function of disjoint sets $B_i \in \Sigma$ and $a_i \in Y$, then we obtain

$$\begin{array}{l} \left\langle \int_D \chi d\eta, \zeta\right\rangle = \sum_{i=1,\dots,k} \langle a_i \eta(B_i), \zeta\rangle = \\ = \sum_{i=1,\dots,k} \int_{B_i} \langle \Upsilon a_i, \zeta\rangle\, d\phi = \int_D \langle \Upsilon\chi, \zeta\rangle\, d\phi \end{array}$$

for all $\zeta \in Z^*$.

For each function $f \in L^1(E, Y, \phi)$, there is a sequence $\{\chi_k\}$ of simple functions such that $\chi_k \overset{k\to\infty}{\longrightarrow} f$ ϕ-almost everywhere and in the topology of $L^1(E, Y, \phi)$. We obtain

$$\int_D \chi_k d\eta \overset{k\to\infty}{\longrightarrow} \int_D f d\eta,$$

$$\left\langle \int_D \chi_k d\eta, \zeta\right\rangle \overset{k\to\infty}{\longrightarrow} \left\langle \int_D f d\eta, \zeta\right\rangle,$$

$$\left\langle \int_D \chi_k d\eta, \zeta\right\rangle = \int_D \langle \Upsilon(\chi_k), \zeta\rangle\, d\mu$$

and

$$\left\langle \int_D f d\eta, \zeta\right\rangle = \int_D \langle \Upsilon(f), \zeta\rangle\, d\mu$$

for all $\zeta \in Z^*$. Thus, we have

$$\eta(D)\xi = \int_D \Upsilon(x)\xi d\mu \in Z$$

so $\eta(D) \in LB(Y, Z)$ for all $D \in \Sigma$.

We assume $\varphi \in L^1(\phi)$ and $\zeta \in Z^*$ and $\eta \neq 0$. The set function ϕ is absolutely continuous with respect to $Var(\mu)$ thus there exists a locally $Var(\mu)$-integrable positive function g such that

$$\int_D |\varphi|\, d\phi = \int_D g\, |\varphi|\, dVar(\mu).$$

The mapping $\Psi(\eta)(\cdot): E \to LB(Y,Z)$ is given by $\Upsilon = \Psi(\eta)\, g\frac{1}{h}$ such that $|h(x)| = 1$, $\mu = hVar(\mu)$ and $Var(\mu) = h^{-1}\mu$ thus $\phi = g\frac{1}{h}\mu$. Since $\|\Psi(\eta)(x)\|_{Y\to Z} \overset{\phi-a.e.}{=} 1$ so that $\|\Upsilon(x)\|_{Y\to Z} = g(x)$ holds μ-almost everywhere since $\|\Psi(\eta)(x)\|_{Y\to Z}\, g(x) = g(x)$, then we take $\phi = \|\Upsilon\|_{Y\to Z} Var(\mu)$ and obtain

$$\int_D |\varphi|\, d\phi = \int_D \|\Upsilon\|_{Y\to Z} |\varphi|\, dVar(\mu)$$

for all $D \in \Sigma$. The theorem has been proven. $\square$

Now, we formulate the third theorem of this section.

Theorem 4.3. *Let μ be a σ-finite scalar measure on Σ. Let $\eta: \Sigma \to LB(Y,Z)$ be a vector measure with $Var(\eta) < \infty$, and let η be absolute continuous with respect to μ. Then, there exists a function $\Psi(\eta)(\cdot): E \to LB(Y,Z)$ such that*

$$\left\langle \int_D f(x)\, d\eta(x), \zeta \right\rangle = \int_D \langle \Psi(\eta)(x) f(x), \zeta\rangle\, d\mu(x) \tag{14}$$

for all $f \in L^1(E, Y, Var(\eta))$ and all $\zeta \in Z^*$ and all $D \in \Sigma$; and

$$\int_D \varphi(x)\, dVar(\eta)(x) = \int_D \|\Psi(\eta)(x)\|_{Y\to Z}\, \varphi(x)\, dVar(\mu)(x) \tag{15}$$

for all $\varphi \in L^1(Var(\eta))$ and all $D \in \Sigma$.

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