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A unique common fixed point result for compatible reciprocal continuous four self -maps in complete metric space

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Abstract: In the present paper, we prove a unique common fixed point theorem for compatible reciprocal continuous for four self- mappings in metric space. Which is an extension, and generalization of results existing in the literature.

Key words: Compatible mapping, common fixed point, metric space, reciprocal continuous.

AMS subject classifications: 47H10, 54H25.

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1. Introduction

In non-linear analysis the study of fixed point theorems are very important , and a very active research . The notion of compatible mappings first introduced by G.Jungck[2] and which is more general than the commuting and weakly commuting mappings it was introduced by S.Sessa [9]. And the concept of compatible mappings of Type (A) given by Jungck. Et.al. [3].The concept of compatible mappings of Type (B) was introduced by Pathak and Khan [4]. In 1998, extension of compatible mappings of Type(A)that is compatible mappings of Type (C). And the concept of reciprocal continuous was introduced by Pant [7] and obtaind some fixed point theorems. Recently, K.Jha. et .al . [1] obtained fixed point results for reciprocal continuous compatible mappings in metric space. In this paper we obtain fixed point results for compatible and reciprocal continuous mappings in metric space. Our result is an extension and generalization of the results of [1].

2. Preliminaries

The following are useful in our main results .

Definition 2.1. [2] The mappings M and N of a metric space (X, ρ) are said to be compatible mappings if and only if $\lim_{n\to\infty} \rho(MNx_n, NMx_n) = 0$ when ever $\{x_n\}$ is a sequence in X such that if $\lim_{n\to\infty} MNx_n = NMx_n = t$ for some $t \in X$

Definition 2.2. [7] The mappings M and N of a metric space (X, ρ) are said to be reciprocally continuous if $\lim_{n\to\infty} MNx_n = M(t)$ and $\lim_{n\to\infty} NMx_n = N(t)$ when ever $\{x_n\}$ is a sequence in X such that if $\lim_{n\to\infty} MNx_n = NMx_n = t$ for some $t \in X$

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Proposition 2.1. *If M and N are compatible and reciprocally continuous mappings on a metric space (X,ρ). Then we have*
(i). $M(t)=N(t)$ where $\{x_n\}$ is a sequence in X such that if $\lim_{n\to\infty} MNx_n = NMx_n = t$ for some $t\in X$.
(ii).If there exists $u\in X$ such that $Mu=Nu=t$, then $MNu=NMu$.

Lemma 2.1. [4] *Let M, N, P and Q be self – mappings of a metric space (X,ρ) such that $MX\subset QX$, $NX\subset PX$. Also assume further that given $\epsilon>0$, there exists a $\delta>0$ such that for all $x,y\in X$, we have $\epsilon<D(x,y)<\epsilon+\delta$ implies $\rho(Mx,Ny)\le\epsilon$ and $\rho(Mx,Ny)<D(x,y)$ when ever $D(x,y)=Max\{[\rho(Px,Qy)+\rho(Mx,Px)+\rho(Qx,Py)]/3,[\rho(Px,Ny)+\rho(Mx,Qy)]/2\}$. Then for each $x_0\in X$, the sequence $\{y_n\}\in X$ defined by the rule $y_{2n}=Mx_{2n}=Qx_{2n+1}$; $y_{2n+1}=Nx_{2n+1}=Px_{2n+2}$ is a Cauchy sequence.*

3. Main Results

In this section, we prove our main theorem.

Theorem 3.1. *Let (M,P) and (N,Q) be compatible and two reciprocally continuous self mappings in complete metric space (X,ρ) such that*

for $\epsilon>0$, there exists a $\delta>0$ for all $x,y\in X$,
we have $\epsilon<D(x,y)<\epsilon+\delta$ implies $\rho(Mx,Ny)\le\epsilon$ and $\rho(Mx,Ny)<D(x,y)$ where

$$D(x,y)=Max\{[\rho(Px,Qy)+\rho(Mx,Px)+\rho(Qx,Py)]/3,[\rho(Px,Ny)+\rho(Mx,Qy)]/2\}. \tag{1}$$

$$\rho(Mx,Ny)<Max\{\lambda_1[\rho(Px,Qy)+\rho(Mx,Px)+\rho(Qx,Py)]/3,\lambda_2[\rho(Px,Ny)+\rho(Mx,Qy)]/2\} \tag{2}$$

$$M(X)\subset Q(X),N(X)\subset P(X) \tag{3}$$

Then M, N, P and Q are having a unique common fixed point.

Proof. Let $x_0\in X$. Define sequences in X, $\{x_n\}$ and $\{y_n\}$. Since $M(X)\subset Q(X)$, $N(X)\subset P(X)$,

$$y_{2n}=Mx_{2n}=Qx_{2n+1},y_{2n+1}=Nx_{2n+1}=Px_{2n+2}. \tag{4}$$

By the above Lemma (2.1) we get that and $\{y_n\}$ is a Cauchy sequence. Since X is complete, so there exists a point z in X such that $y_n\to z$. From (1) we get that,

$$y_{2n}=Mx_{2n}=Qx_{2n+1}\to z;y_{2n+1}=Nx_{2n+1}=Px_{2n+2}\to z. \tag{5}$$

Since (M,P) is compatible and reciprocal continuous by proposition, we get that $Mz=Pz$.
Claim: $Mz=z$. If $Mz\ne z$ then from (2) we get that
$\rho(Mz,Nx_{2n+1})$
$<\max\{\lambda_1[\rho(Pz,Qx_{2n+1})+\rho(Mz,Pz)+\rho(Nx_{2n+1},Qx_{2n+1})]/3,\lambda_2[\rho(Pz,Nx_{2n+1})+\rho(Mz,Qx_{2n+1})]/2\}$
Letting n $\to\infty$, we get that
$\rho(Mz,z)<Max\{\lambda_1[\rho(Pz,z)+\rho(Mz,Pz)+\rho(z,z)]/3,\lambda_2[\rho(Pz,z)+\rho(Mz,z)]/2\}$

$=$ Max $\{\lambda_1[\rho(Pz,z),\lambda_2\rho(Mz,z)\}$

$<\rho(Mz,z)$, which is a contradiction.

Therefore, we get that $Mz = z$, and therefore

$$Mz = Pz = z. \tag{6}$$

$$\text{Hence } z \text{ is a common fixed point of } M \text{ and } P. \tag{7}$$

Since $MX \subset QX$ there exists a point u in X such that $Mz = Qu$.
Claim: $Nu = Qu$. If $Nu \neq Qu$ then from then from (2) we get that
$\rho(Qu, Nu) = \rho(Mz, Nu)$,

$$< Max\{\lambda_1[\rho(Pz, Qu) + \rho(Mz, Pz) + \rho(Nu, Qu)]/3, \lambda_2[\rho(Pz, Nu) + \rho(Mz, Qu)]/2\}$$

$$< Max\{\lambda_1[\rho(Qu, Qu) + \rho(Pz, Pz) + \rho(Nu, Qu]/3, \lambda_2[\rho(Qu, Nu) + \rho(Qu, Qu)]/2\}$$

$< \rho(Qu, Nu)$, which is a contradiction.
Therefore, we get that $Qu = Nu$ and therefore

$$Mz = Pz = Qu = Nu = z. \tag{8}$$

By the Proposition (2.1), we get that

$$NNU = QNu. \tag{9}$$

Moreover, we get that

$$NNu = NQu = QQu = QNu. \tag{10}$$

Claim: $NNu = Nu$. If $NNu \neq Nu$ then from then from (2) we get that
ρ(Nu, NNu) = $\rho(Mz, NNu)$,

$$< Max\{\lambda_1[\rho(Pz, QNu) + \rho(Mz, Pz) + \rho(NNu, QNu)]/3, \lambda_2[\rho(Pz, NNu) + \rho(Mz, QNu)]/2\}$$

$$< Max\{\lambda_1[\rho(Nu, NNu) + \rho(Mz, Mz) + \rho(NNu, NNu)]/3, \lambda_2[\rho(Nu, NNu) + \rho(Nu, NNu)]/2\}$$

$$< Max\{\lambda_1[\rho(Nu, NNu)]/3, \lambda_2[\rho(Nu, NNu)]\}$$

$< \rho(Nu, NNu)$, which is a contradiction.

Therefore, we get that $NNu = Nu$.
From (10) we get that $NNu = QNu = Nu$.
That implies, we have $Nu = Mz = z$. Moreover, $NNu = QNu$.
Implies that, $Nz = Qz = z$. Therefore, z is a common fixed point of N and Q.
Hence, z is the common fixed point of M, N, P and Q.
Uniqueness: Let $z_1 \neq z$ is another fixed point of M, N, P and Q.
Then we get that $Mz_1 = Nz_1 = Pz_1 = Qz_1 = z_1$.
By (2) we get that,
$\rho(Mz, Nz_1) <$ $\max\{\lambda_1[\rho(Nz, Qz_1) + \rho(Mz, Pz) + \rho(Nz_1, Qz_1)]/3, \lambda_2[\rho(Pz, Nz_1) + \rho(Mz, Qz_1)]frm{-}e\}$,

$< \rho(Mu, Nz_1)$,
which is a contradiction.

Therefore, M, N, P and Q are having a unique common fixed point. This completes the proof of the theorem. □

4. Conclusion

Our results are proved in this paper are more general than the results of [1] .

Conflict of interest: The author has declared there is no conflict of interest.

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