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Multi fuzzifying topology

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Abstract: In this paper, Characterizations of stratified and transitive L -topologies and multi-fuzzifying topology are introduced. First we introduce the concepts of stratified and transitive L -co-topology and stratified and transitive L -closure operator to characterize the concept of stratified and transitive L -topology, where L is a complete MV -algebra. Second, we introduce the concepts of multi-fuzzifying topology, L -fuzzifying co-neighborhood system, L -fuzzifying co-contiguity, L -fuzzifying interior operator and L -fuzzifying closure operator to characterize multi-fuzzifying topology, where L is a completely distributive complete residuated lattice satisfying the double negation.law, Characterizations of stratified and transitive L -topology are introduced where L is a complete MV-algebra, Characterizations of multi-fuzzifying topology are introduced where L is a completely distributive complete residuated lattice in which the double negation law is satisfied.

Key words: Multi Fuzzifying Topology; L -fuzzifying co-contiguity, MV-algebra

1. Introduction

Since the theory of fuzzy sets was introduced by [Zadeh (13)], many topological notions were introduced and discussed in fuzzy setting. In [Chang (2)], introduced and studied the notion of fuzzy topology as a crisp subset of the family of fuzzy subsets of an ordinary set. Also, [Lowen (7)], [Hutton (6)], [Pu and Liu (9)], and [Wong (10, 11)], discussed respectively various aspects of fuzzy topology where in their approaches the fuzzy topology was defined also as a crisp subset of the fuzzy power set of a nonempty set. [Goguen(3)], introduced the concept of L -set (L - fuzzy set) as a generalization of the concept of fuzzy set where L is some type of lattice. It is worth to mention that [Höhle (5)] used L as a complete MV -algebra but [Ying (12)] used L as a complete residuated lattice. The concept of complete residuated lattice was introduced by [Pavelka (8)]. The concept of L -fuzzifying topology appeared in [Höhle (4)] under the name "L -fuzzy topology" (cf. Definition 4.6, Proposition 4.11 in [Höhle (4)] where L is a completely distributive complete lattice. In the case of $L = [0, 1]$ this terminology traces back to [Ying (12)] studied the fuzzifying topology and elementarily developed fuzzy topology from a new direction with semantic method of continuous valued logic. Fuzzifying topology (resp. L -Fuzzifying topology) in the sense of M. S. Ying (resp. U. Höhle) was introduced as a fuzzy subset (resp. an L -Fuzzy subset) of the power set of an ordinary set. [Höhle (5)] introduced and studied a characterization of stratified and transitive L - topology by stratified and transitive L -interior operator K, where L is a complete MV-algebra. A characterization of L -fuzzifying topology by L -fuzzifying neighborhood system, where L is a completely distributive, was given also in [Höhle (5)]. Finally, [Höhle (5)] introduced a

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characterizations of stratified and transitive L - topology by L -contiguity and L -fuzzifying topology, where L is a completely distributive complete MV-algebra. In this paper we introduce and discuss a generalization of a result introduced by [Höhle (5)] in L -fuzzifying topology. Mainly we add Characterizations of stratified and transitive L-topology and multi fuzzifying topology. In Section 1, Lattice theory and basic concepts are introduced. In Section 2, Characterizations of multi fuzzifying topology are introduced where L is a completely distributive complete residuated lattice in which the double negation law is satisfied. In Section 3, L- fuzzifying neighborhood and L- fuzzifying closure operations are considered.

1.1. Lattice theory and basic concepts

Definition 1.1. **(**Höhle (5)]**).** The double negation law in a complete residuated lattice L is given as follows: $\forall a, b \in L,\ (a \to \bot) \to \bot = a$.

Definition 1.2. **(**Höhle (5)**).** A structure $(L, \vee, \wedge, *, \to, \bot, \top)$ is called a strictly two-sided commutative quantale iff

(1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice whose greatest and least element are $\top, \bot$ respectively,

(2) $(L, *, \top)$ is a commutative monoid,

(3)(a) $*$ is distributive over arbitrary joins, i.e.,

$a * \bigvee_{j \in J} b_j = \bigvee_{j \in J} (a * b_j)\ \forall a \in L,\ \forall \{b_j \mid j \in J\} \subseteq L,$

(b) $\to$ is a binary operation on L defined by :

$a \to b = \bigvee_{\lambda * a \le b} \lambda \quad \forall a, b \in L.$

Definition 1.3. **(**Höhle (5)). A structure $(L, \vee, \wedge, *, \to, \bot, \top)$ is called a complete MV- algebra iff the following conditions are satisfied:

(1) $(L, \vee, \wedge, *, \to, \bot, \top)$ is a strictly two-sided commutative quantale,

(2) $\forall a, b \in L,\ (a \to b) \to b = a \vee b.$

Definition 1.4. (Pavelka (8), Ying (12)). A structure $(L, \vee, \wedge, *, \to, \bot, \top)$ is called a complete residuated lattice iff

(1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice whose greatest and least element are $\top, \bot$ respectively,

(2) $(L, *, \top)$ is a commutative monoid, i.e.,

(a) $*$ is a commutative and associative binary operation on L, and

(b) $\forall\ a \in L, a * \top = \top *\ a = a,$

(3)(a) $*$ is isotone,

(b) $\to$ is a binary operation on L which is antitone in the first and isotone in the second variable,

(c) $\rightarrow$ is couple with $*$ as: $a * b \le c$ iff $a \le b \rightarrow c \ \ \forall \ a, b, c \in L$.

Definition 1.5. (Birkhoff (1)). Let L be a complete lattice. We say that $\wedge$ is distributive over arbitrary joins iff

$$\forall\{\alpha_j : j \in J\} \text{ and } \forall \alpha \in L, \ \alpha \wedge (\bigvee_{j \in J} \alpha_j) = \bigvee_{j \in J} (\alpha \wedge \alpha_j).$$

Corollary 1.1. $(L, \vee, \wedge, *, \rightarrow, \bot, \top)$ is a complete MV- algebra iff $(L, \vee, \wedge, *, \rightarrow, \bot, \top)$ is a complete residuated lattice satisfies the additional property (MV) $(a \rightarrow b) \rightarrow b = a \vee b \ \ \forall a, b \in L$.

Definition 1.6. (Höhle (5)). A structure $(L, \vee, \wedge, *, \rightarrow, \bot, \top)$ is called a complete MV- algebra iff the following conditions are satisfied:

(1) $(L, \vee, \wedge, *, \rightarrow, \bot, \top)$ is a strictly two-sided commutative quantale,

(2) $\forall a, b \in L, \ (a \rightarrow b) \rightarrow b = a \vee b$.

Theorem 1.1 (Corollary 3.15 [Höhle (5)). Let $(L, \le, *)$ be a complete $MV-$algebra and $\odot = \wedge$. Furthermore let $(L, \le)$ be a completely distributive lattice complete MV-algebra. Then L-fuzzifying topologies, L-fuzzy contiguity relations and stratified and transitive L-topologies are equivalent concepts.

2. Multi Fuzzifying Topology

Definition 2.1. Let X be a nonempty set and. $[X]^m$ is the set of all msets whose elements are in X such that no element in the mset occurs more than m times. The power mset $P\ ([X]^m)$ of X is the set of all sub msets of X. An element $\mathcal{T} : P\ ([X]^m) \rightarrow I$ is called an multi- fuzzifying topology on X iff it satisfies the following axioms:

(1) $\mathcal{T}(X) = \mathcal{T}(\phi) = 1$,

(2) $\forall A, B \in P\ ([X]^m), \mathcal{T}(A \cap B) \ge \mathcal{T}(A) \wedge \mathcal{T}(B)$,

(3) $\forall \{A_j \,|j \in J\} \subseteq P\ ([X]^m), \ \mathcal{T}(\bigcup_{j \in J} A_j) \ge \bigwedge_{j \in J} \mathcal{T}(A_j)$. The pair $(X, \mathcal{T})$ is called an multi- fuzzifying topological space.

Example 2.1. Let X = {x, x, y, y}, then $[X]^m = \{\frac{2}{x}, \frac{2}{y}\}$

and $P\ ([X]^m) = \{\phi, \frac{2}{x}, \frac{2}{y}, \frac{1}{\{x\}_2}, \frac{1}{\{y\}_2}, \frac{4}{\{x,y\}}, \frac{2}{\{x,y\}_{2,1}}, \frac{2}{\{x,y\}_{1,2}}, \frac{1}{\{x,y\}_{2,2}}\}$. Define,

$$\mathcal{T}(A) = \begin{cases} 1 & A = \phi, \frac{1}{\{x,y\}_{2,2}} \\ \frac{1}{2} & A = \frac{2}{x}, \frac{2}{y}, \frac{1}{\{x\}_2}, \frac{1}{\{y\}_2} \\ \frac{1}{3} & A = \frac{4}{\{x,y\}}, \frac{2}{\{x,y\}_{2,1}}, \frac{2}{\{x,y\}_{1,2}} \end{cases}$$

Then $\mathcal{T}$ is multi- fuzzifying topology on X.

Remark 2.1 when $m=1$, multi- fuzzifying topology $\Rightarrow$ fuzzifying topology.

Definition 2.2. Let X be a nonempty set. A map $\mathcal{F}: P\ ([X]^m) \rightarrow I$ is called an multi-fuzzifying co-topology (or the family of multi-fuzzifying closed sets) if it satisfies the following axioms:

(1) $\mathcal{F}(X)=\mathcal{F}(\phi)=1$,

(2) $\forall\, A, B \in P\ ([X]^m),\ \mathcal{F}(A \cup B) \geq \mathcal{F}(A) \wedge \mathcal{F}(B)$,

(3) $\forall\, \{A_{j|j\in J}\} \subseteq P\ ([X]^m),\ \mathcal{F}(\cap_{j\in J} A_j) \geq \wedge_{j\in J} \mathcal{F}(A_j)$.

Definition 2.3 The family of multi- fuzzifying closed sets, denoted by $F \in I^{P\ ([X]^m)}$, is defined as follows: $F(A)=\tau(X-A)$ where $X-A$ is the complement of A.

Definition 2.4 Let (X,τ) be an multi- fuzzifying topological space and $A \subseteq X$, Let $x \in X$. The multi-fuzzifying neighbourhood system of x, denoted by $N_x \in I^{P\ ([X]^m)}$, is defined as follows:

$$N_x(A)=\bigvee_{x\in B\subseteq A} \tau(B).$$

Proposition 2.1. Let (X,τ) be an multi-fuzzifying topological space and let $A, B \in P\ ([X]^m)$. Then $\forall x \in X$,

(1) $N_x(X)=\top,\quad N_x(\phi)=\bot$,

(2) $A \subseteq B \Rightarrow N_x(A) \leq N_x(B)$,

(3) If $\wedge$ is distributive over arbitrary joins.

Then $N_x(A \cap B)=N_x(A) \wedge N_x(B)$,

(4) $N_x(A) \leq \bigvee_{y\in X-B}(N_y(A) \vee N_x(B))\ \ \forall B \in P\ ([X]^m)$.

Proof. (1)(a) $N_x(X)=\bigvee_{x\in B\subseteq X} \tau(B)=\top$ because $\tau(X)=\top$.

(1)(b) $N_x(\phi)=\bigvee_{x\in H\subseteq \phi} \tau(H)=\bot$.

(2) $N_x(A)=\bigvee_{x\in H\subseteq A} \tau(H) \leq \bigvee_{x\in H\subseteq B} \tau(H)=N_x(B)$.

(3) From (2), we have $N_x(A \cap B) \leq N_x(A) \wedge N_x(B)$, and

$$\begin{aligned} N_x(A\cap B) &= \bigvee_{x\in H\subseteq A\cap B} \tau(H)=\bigvee_{x\in H_1\cap H_2\subseteq A\cap B} \tau(H_1\cap H_2) \geq \bigvee_{x\in H_1\subseteq A,\ x\in H_2\subseteq B} \tau(H_1\cap H_2) \\ &\geq \bigvee_{x\in H_1\subseteq A,\ x\in H_2\subseteq B}(\tau(H_1)\wedge\tau(H_2)) \geq \bigvee_{x\in H_1\subseteq A,\ x\in H_2\subseteq B}(\tau(H_1)\wedge\tau(H_2)) \\ &= \bigvee_{x\in H_1\subseteq A} \tau(H_1) \wedge \bigvee_{x\in H_2\subseteq B} \tau(H_2)=\varphi_x(A)\wedge\varphi_x(B). \end{aligned}$$

(4) Let x be a fixed point in X and let A, G be subsets of X s.t. $x \in G \subseteq A$. Now,

(a) for every $B \in P(X)$ s.t. $G \cap (X-B) \neq \phi$, there exists $y_0 \in G \cap (X-B)$.

Now, $N_{y_0}(A) = \bigvee_{y_0 \in H \subseteq A} \tau(H) \geq \tau(G)$. Hence $\bigvee_{y \in X-B} N_y(A) \geq N_{y_0}(A) \geq \tau(G)$. And

(b) for every $B \in P(X)$ s.t. $G \cap (X-B) = \phi$, $N_x(B) = \bigvee_{x \in M \subseteq A \cap B} \tau(M) \geq \tau(G)$.

Hence $(\bigvee_{y \in X-B} N_y(A)) \vee N_x(B) \geq \bigvee_{x \in G \subseteq A} \tau(G) = N_x(A) \quad \forall B \in P\ ([X]^m)$.

Corollary 2.1 $\bigwedge_{x \in A} N_x(A) = \tau(A)$.

Theorem 2.1 For any $x \in X,\ A \in P\ ([X]^m)$, $\bigwedge_{x \in A} \bigvee_{x \in B \subseteq A} \tau(B) = \tau(A)$

Definition 2.5. The multi-fuzzifying derived set $d_\tau(A)$ of A is defined as follows:

$$d_\tau(A)(x) = \bigwedge_{B \cap (A - \{x\}) = \phi} (1 - N_x(B)).$$

Lemma 2.1. $d_\tau(A)(x) = 1 - N_x((X-A) \cup \{x\})$.

Definition 2.6. The multi-fuzziyfying closure $\overline{A} \in I^X$ of $A \in P\ ([X]^m)$ is defined as follows:

$$\overline{A}(x) = \bigwedge_{x \notin B \supseteq A} (1 - F(B)).$$

Lemma 2.1. $\overline{A}(x) = 1 - N_x(X - A)$.

Definition 2.7. For any $A \subseteq X$, the interior $A^\circ \in I^X$ of A, is defined as follows: $A^\circ(x) = N_x(A)$.

3. On multi-fuzzifying neighbourhoods and multi-fuzzifying closure operations

Definition 3.1. Let X be a nonempty set. A map $(\ )^- :\ P\ ([X]^m) \to L^X$ is called an multi-fuzzifying closure operator if $(\ )^-$ satisfies the following conditions:

(1^-) $(\phi)^- = 1_\phi$,

(2^-) $(A \cup B)^- = (A)^- \vee (B)^-$,

(3^-) $A \leq (A)^-$,

(4^-) $(A)^-(x) \leq \bigwedge_{y \notin B} ((A)^-(y) \wedge (B)^-(x))$.

The multi-fuzzifying closure operator $(\ )^-$ defined in Definition 3.1 induced by the multi-fuzzifying topology τ will be denoted in this Section by cl_τ and one can have it as follows:

$$cl_\tau(A)(x) = N_x(X - A) \to \perp \ \forall x \in X.$$

Remark 3.1. One can observe that the statements in Proposition 2.1 are generalizations of the corresponding statements in Proposition 2.12 (Höhle, (1999)[22]). Note that if L satisfies the completely distributive law, then it satisfies that $\wedge$ is distributive over arbitrary joins.

Proposition 3.1. Let (X, τ) be an multi-fuzzifying topological space. Then:

(1) If L satisfies the double negation law, then $N_x(A) = cl_\tau(X - A)(x) \to \bot$,

(3) $A \leq cl_\tau(A)$,

(4) $A \subseteq B \Rightarrow cl_\tau(A) \leq cl_\tau(B)$,

(5) $cl_\tau(A \cup B) = cl_\tau(A) \vee cl_\tau(B)$.

Proof. (1) Since $cl_\tau(X - A)(x) = N_x(A) \to \bot$, then

$$\varphi_x(A) = (\varphi_x(A) \to \bot) \to \bot = cl_\tau(X - A)(x) \to \bot.$$

(2) $cl_\tau(\phi)(x) = \varphi_x(X) \to \bot = \top \to \bot = \bot \ \forall x \in X$. Then $cl_\tau(\phi) = 1_\phi$.

(3) If $x \in A$, then $cl_\tau(A)(x) = N_x(X - A) \to \bot = \bot \to \bot = \top = A(x)$.

If $x \notin A$, then $cl_\tau(A)(x) \geq A(x)$. Hence $cl_\tau(A) \geq A$.

(4) Let $A \subseteq B$. Then $cl_\tau(A)(x) = N_x(X - A) \to \bot \leq N_x(X - B) \to \bot = cl_\tau(B)(x)$.

(5)

$$\begin{aligned} cl_\tau(A \cup B) &= N_x(X - (A \cup B)) \to \bot = N_x((X - A) \cap (X - B)) \to \bot \\ &= (N_x(X - A) \wedge N_x(X - B)) \to \bot = (N_x(X - A) \to \bot) \vee (N_x(X - B) \to \bot) = cl_\tau(A) \vee cl_\tau(B). \end{aligned}$$

Definition 3.2. A stratified and transitive L-topology τ on a nonempty set X is a subset of L^X satisfies the following conditions:

(01) $1_X,\ 1_\phi \in \tau$,

(02) $g_1, g_2 \in \tau \Rightarrow g_1 \wedge g_2 \in \tau$,

(03) $\{g_j \mid j \in J\} \subseteq \tau \Rightarrow \bigvee_{j \in J} g_j \in \tau$,

$(\sum 1)$ $g \in \tau,\ a \in L \Rightarrow (a.1_X) * g \in \tau$, (Truncation Condition)

$(T1)$ $g \in \tau,\ a \in L \Rightarrow (a.1_X) \vee g \in \tau$, (Translation-invariance)

$(T2)$ $g \in \tau,\ a \in L \Rightarrow g \rhd (a.1_X) \in \tau$. (Co-Stratification)

The pair (X, τ) is called a stratified and transitive L-topological space.

Definition 3.3. Let X be a nonempty set. A map $K : L^X \to L^X$ is called a stratified and transitive interior operator if K satisfies the following conditions:

(K_0) $K(1_X) = 1_X$,

(K_1) $f \leq g \Rightarrow K(f) \leq K(g)$,

(K_2) $K(f) \wedge K(g) \leq K(f \wedge g)$,

(K_3) $K(f) \leq f$,

(K_4) $K(f) \leq K(K(f))$,

(K_5) $(a.1_X) * K(f) \leq K((a.1_X) * f)$,

(K_6) $(a.1_X) \vee K(f) = K((a.1_X) \vee f)$.

Definition 3.4. Let X be a nonempty set. An element $c \in L^{X \times P\ ([X]^m)}$ is called an L-fuzzy contiguity relation on X iff c fulfills the following axioms:

(c_1) $c(x, \phi) = \bot\ \ \forall\, x \in X$.

(c_2) $c(x, A \cup B) = c(x, A) \vee c(x, B)$, (Distributivity),

(c_3) $c(x, A) = \top$, whenever $x \in A$,

(c_4) $(\bigwedge_{y \in B} c(y, A)) \wedge c(x, B) \leq c(x, A)$. (Transitivity).

Theorem 3.1. If L is a complete MV-algebra, then the concept of stratified and transitive L-topology, and the concept of stratified and transitive L- interior operator are equivalent notions.

Theorem 3.2 (Proposition 3.13 [5]). Let (X, τ) be an multi-fuzzifying topological space, and let L satisfies the completely distributive law then the L-fuzzifying neighborhood system $(N_x)_{x \in X}$ satisfies the following conditions:

(f_1) $N_x(X) = \top$, $\forall\, x \in X$, (Boundary conditions)

(f_2) $N_x(A \cap B) = N_x(A) \wedge N_x(B)$, (Intersection property)

(u_3) $N_x(A) = \bot$ whenever $x \notin A$

(u_4) $N_x(A) \leq \bigvee_{y \notin B}(N_y(A) \vee N_x(B))\ \forall B \in P(X)$. Furthermore $\tau(A) = \bigwedge_{x \in A} \varphi_x(A)\ \ \forall A \in P\ ([X]^m)$.

Theorem 3.3 (Proposition 3.14 [5]). let L satisfies the completely distributive law and Let $(N_x)_{x \in X}$ be a system satisfies the properties (f_1), (f_2), (u_3), (u_4) in Theorem 3.2 above . Then $(N_x)_{x \in X}$ induces an multi-fuzzifying topology τ on X by $\tau(A) = \bigwedge_{x\in A} N_x(A)\ \ \forall A \in P\ ([X]^m)$. Moreover the following formula holds $N_x(A) = \bigvee_{x\in B\subseteq A} \tau(B)$.

Theorem 1.4.4 (Corollary 3.15 [Höhle (5)). Let $(L, \leq, *)$ be a complete $MV-$algebra and $\odot = \wedge$. Furthermore let $(L, \leq)$ be a completely distributive lattice complete MV-algebra. Then multi-fuzzifying topologies, L-fuzzy contiguity relations and stratified and transitive L-topologies are equivalent concepts.

3.2. Some Characterizations of stratified and transitive L-topology

In this section L is assumed to be a complete MV-algebra.

Definition 3.2.1. Define the binary operator $\circledast$ on L is defined as follows:

$$\alpha \circledast \beta = ((\alpha \to \bot) * (\beta \to \bot)) \to \bot$$

Definition 3.2.2. Define the binary operator $\underset{\to}{\triangle}$ on L is defined as follows:

$$\alpha \underset{\to}{\triangle} \beta = \bigvee_{\xi\in L,\ \xi\wedge\beta\leq\alpha} \xi.$$

Definition 3.2.3. A stratified and transitive L-co-topology F on a nonempty set X is a subset of L^X satisfies the following conditions:

$(co-01)$ $1_X, 1_\phi \in F$,

$(co-02)$ $g_1, g_2 \in F \Rightarrow g_1 \vee g_2 \in F$,

$(co-03)$ $\{g_{j|j\in J}\} \subseteq F \Rightarrow \bigwedge_{j\in J} g_j \in F$,

$(co-\sum 1)$ $g \in F,\ \alpha \in L \Rightarrow (\alpha.1_X) \circledast g \in F$,

$(co-T_2)$ $g \in F,\ \alpha \in L \Rightarrow g \underset{\to}{\triangle} \alpha.1_X \in F$.

By making use of the concept of stratified and transitive L-co-topology we characterize the stratified and transitive L-topology.

Theorem 3.2.1. Let τ be a stratified and transitive L-topology on X. Define $F_\tau : P\ ([X]^m) \to L$ as: $F_\tau = \{g \in L^X\, |(g \to \bot) \in \tau\}$. Then F_τ is a stransitive and transitive L-co-topology on X induces by a stratified and transitive L-topology τ on X. Let F be a stransitive and transitive L-co-topology on X. Define $\tau_F : P\ ([X]^m) \to L$ as: $\tau_F = \{g \in L^X\, |(g \to \bot) \in F\}$. Then τ_F is a stratified and transitive L-topology on X induces by a stransitive and transitive L-co-topology F on X. Furthermore $\tau_{F_\tau} = \tau$ and $F_{\tau_F} = F$.

Proof. (A) $(co-01)$ $1_X \in \tau \Rightarrow (1_X \to \bot) = 1_\phi \in F_\tau$ and $1_\phi \in \tau \Rightarrow (1_\phi \to \bot) = 1_X \in F_\tau$.

$(co-02)$ $g_1, g_2 \in F_\tau \Rightarrow (g_1 \to \bot), (g_2 \to \bot) \in \tau \Rightarrow$ $(g_1 \to \bot) \wedge (g_2 \to \bot) = (g_1 \vee g_2) \to \bot \in \tau$

$\Rightarrow (g_1 \vee g_2) \in F_\tau$.

$(co-03)$ $\{g_{j|j\in J}\} \subseteq F_\tau \Rightarrow \{g_j \to \bot \,|j \in J\} \subseteq \tau \Rightarrow \bigvee\limits_{j\in J}(g_j \to \bot) = \bigwedge\limits_{j\in J} g_j \to \bot \in \tau \Rightarrow \bigwedge\limits_{j\in J} g_j \in F_\tau$.

$(co-\sum 1)$ $g \in F_\tau, \alpha \in L \Rightarrow (g \to \bot) \in \tau, (\alpha \to \bot) \in L \Rightarrow ((\alpha \to \bot).1_X) * (g \to \bot) \in \tau$

$\Rightarrow (((\alpha \to \bot).1_X) * (g \to \bot)) \to \bot \in F_\tau \Rightarrow (\alpha.1_X) \circledast g \in F_\tau$.

$(co-T_2)$ $g \in F_\tau,\ \alpha \in L \Rightarrow (g \to \bot) \in \tau,\ \alpha \in L$

$\Rightarrow (g \to \bot) \rhd ((\alpha \to \bot).1_X) \in \tau \Rightarrow \left(\bigwedge_{\lambda\in L,\ \lambda\vee(\alpha\to\bot).1_X\ \geq\ g\to\bot} \lambda\right) \to \bot$

$= \bigvee_{(\lambda\to\bot)\in L,(\lambda\to\bot)\wedge\alpha.1_X\ \leq\ g} (\lambda \to \bot) = g \underset{\to}{\wedge} \alpha.1_X \in F_\tau$.

(B) $(O1)$ $1_X \in F \Rightarrow (1_X \to \bot) = 1_\phi \in \tau_F$ and $1_\phi \in F \Rightarrow (1_\phi \to \bot) = 1_X \in \tau_F$.

$(O2)$ Let $g_1, g_2 \in \tau_F \Rightarrow (g_1 \to \bot), (g_2 \to \bot) \in F \Rightarrow (g_1 \to \bot) \vee (g_2 \to \bot) \in F$

$\Rightarrow (g_1 \wedge g_2) \to \bot \in F \Rightarrow (g_1 \wedge g_2) \in \tau_F$.

$(O3)$ $\{g_{j|j\in J}\} \subseteq \tau_F \Rightarrow \{g_j \to \bot \,|j \in J\} \subseteq F \Rightarrow \bigwedge_{j\in J}(g_j \to \bot) \in F = \bigvee_{j\in J} g_j \to \bot \in F$
$\Rightarrow \bigvee_{j\in J} g_j \in \tau_F$.

$(\sum 1)$ $g \in \tau_F,\ \alpha \in L \Rightarrow (g \to \bot) \in F,\ (\alpha \to \bot) \in L \Rightarrow ((\alpha \to \bot).1_X) \circledast (g \to \bot) \in F$

$\Rightarrow ((\alpha \to \bot).1_X) \circledast (g \to \bot)) \to \bot \in \tau_F \Rightarrow ((\alpha.1_X) * g) \in \tau_F$.

$(T2)$ $g \in \tau_F,\ \alpha \in L \Rightarrow (g \to \bot) \in F,\ (\alpha \to \bot) \in L \Rightarrow (g \to \bot) \underset{\to}{\wedge} ((\alpha \to \bot).1_X) \in F$

$\Rightarrow ((g \to \bot) \underset{\to}{\wedge} ((\alpha \to \bot).1_X)) \to \bot \in \tau_F \Rightarrow \left(\bigvee_{\lambda\in L,\lambda\ \wedge\ (\alpha\to\bot).1_X\ \leq\ g\to\bot} \lambda\right) \to \bot \in \tau_F$

$\Rightarrow \left(\bigwedge_{(\lambda\to\bot)\in L,(\lambda\to\bot)\ \vee\ \alpha.1_X\ \geq\ g}(\lambda \to \bot)\right) \in \tau_F \Rightarrow \bigwedge_{\lambda\in L,\lambda\ \vee\ \alpha.1_X\ \geq\ g} \lambda \in \tau_F \Rightarrow g \rhd (\alpha.1_X) \in \tau_F$.

(C) $\tau_{F_\tau} = \{g \in L^X \,|(g \to \bot) \in F_\tau\} = \{g \in L^X \,|(g \in \tau\} = \tau$ and $F_{\tau_F} = \{g \in L^X \,|(g \to \bot) \in \tau_F\}$

$= \{g \in L^X \,|(g \in F\} = F$.

Remark 3.2.1. (1) From conditions (01) and $(\sum 1)$ in Definition 1.4.3 one can deduce that $\forall\alpha \in L,\ \alpha.1_X \in \tau$ (Indeed, from (01) we have that $1_X \in \tau$ so that $\forall\alpha \in L$, one can have from $(\sum 1)$ that $\alpha.1_X * 1_X \in \tau$, i.e., $\alpha.1_X \in \tau$ because $\alpha.1_X * 1_X = \alpha.1_X$.).

(2) We note that Condition (T_1) in Definition 1.4.3 can be obtained from conditions (01), (03) and $(\sum 1)$ (Indeed, from (01) and $(\sum 1)$ we have as above that $\forall \alpha \in L, \quad \alpha.1_X \in \tau$. Now let $g \in \tau$ so that from (03), we have $(\alpha.1_X) \vee g \in \tau$.).

Definition 3.2.4. Let X be a nonempty set. An element $C \in (L^X)^{L^X}$ is called a stratified and transitive L-closure operator on X if C satisfies the following conditions:

(C0) $C(1_\phi) = 1_\phi$,

(C1) $f \leq g \Rightarrow C(f) \leq C(g)$,

(C2) $C(f) \vee C(g) \geq C(f \vee g)$,

(C3) $C(f) \geq f$,

(C4) $C(f) \geq C(C(f))$,

(C5) $(\alpha.1_X) \circledast C(f) \geq C((\alpha.1_X) \circledast f)$,

(C6) $(\alpha.1_X) \wedge C(f) = C((\alpha.1_X) \wedge f)$.

The second characterization of stratified and transitive L-topology in this section is given now by making use of stratified and transitive L-closure operator.

Theorem 3.2.2. Let K be a stratified and transitive L-interior operator. Define $C_K : L^X \to L^X$ as: $C_K(f) = K(f \to \bot) \to \bot$. Then C_K is a stratified and transitive L-closure operator on X induces by a stratified and transitive L-interior operator K on X. Let C be a stratified and transitive L-closure operator on X. Define $K_C : L^X \to L^X$ as: $K_C(f) = C(f \to \bot) \to \bot$. Then K_C is a stratified and transitive L-interior operator on X induces by a stratified and transitive L-closure operator C on X Furthermore $K_{C_K} = K$ and $C_{K_C} = C$.

Proof. (A) $(C0)$ $C_K(1_\phi) \quad = \quad K(1_\phi \to \bot) \to \bot = K(1_X) \to \bot = 1_X \to \bot = 1_\phi$

$(C1)$ Let $f \leq g$. So, $C_K(f) \quad = \quad K(f \to \bot) \to \bot \leq K(g \to \bot) \to \bot = C_K g$

$$
\begin{aligned}
(C2)\ C_K(f) \vee C_K(g) &= (K(f \to \bot) \to \bot) \vee (K(g \to \bot) \to \bot) = (K(f \to \bot) \wedge K(g \to \bot)) \to \bot \\
&\geq K((f \to \bot) \wedge (g \to \bot)) \to \bot \\
&= K((f \vee g) \to \bot)) \to \bot = C_K(f \vee g),
\end{aligned}
$$

$(C3)$ $C_K(f) \quad = \quad K(f \to \bot) \to \bot \geq (f \to \bot) \to \bot = f,$

$(C4)$ $C_K(f) \quad = \quad K(f \to \bot) \to \bot \geq K(K(f \to \bot)) \to \bot = K((C_K(f) \to \bot)) \to \bot = C_K(C_K(f)),$

$$(C5)\ C_K((\alpha.1_X)\circledast f) = K(((\alpha.1_X)\circledast f)\to\bot)\to\bot = K((\alpha.1_X)\to\bot)*(f\to\bot))\to\bot$$
$$\leq ((\alpha.1_X)\to\bot * K(f\to\bot))\to\bot = (\alpha.1_X)\to\bot*(C_K(f)\to\bot))\to\bot$$
$$= \alpha.1_X\circledast C_K(f),$$

$$\begin{aligned}(C6)\ C_K((\alpha.1_X)\wedge f) &= K(((\alpha.1_X)\wedge f)\to\bot)\to\bot = K(((\alpha.1_X)\to\bot)\vee(f\to\bot))\to\bot\\ &= (((\alpha.1_X)\to\bot)\vee K(f\to\bot))\to\bot = (\alpha.1_X)\wedge C_K(f)\end{aligned}$$

(B) $(K0)\ K_C(1_X) \quad = \quad C(1_X\to\bot)\to\bot = C(1_\phi)\to\bot = 1_\phi\to\bot = 1_X,$

$$(K1)\ f\leq g\Rightarrow g\to\bot\leq f\to\bot = C(g\to\bot)\leq C(f\to\bot)$$
$$\Rightarrow C(g\to\bot)\to\bot\geq C(f\to\bot)\to\bot\Rightarrow K_C(f)\leq K_C(g)$$

$$\begin{aligned}(K2)\ K_C(f)\wedge K_C(g) &= (C(f\to\bot)\to\bot)\wedge(C(g\to\bot)\to\bot) = (C(f\to\bot)\vee C(g\to\bot))\to\bot\\ &\leq C((f\to\bot)\vee(g\to\bot))\to\bot = C((f\wedge g)\to\bot)\to\bot = K_C(f\wedge g),\end{aligned}$$

$$(K3)\ K_C(f) \quad = \quad C(f\to\bot)\to\bot\leq(f\to\bot)\to\bot = f,$$

$$(K4)\ K_C(f) \quad = \quad C(f\to\bot)\to\bot\leq C(C(f\to\bot))\to\bot = C(K_C(f\to\bot)\to\bot) = K_C(K_C(f)),$$

$$\begin{aligned}(K5)\ (\alpha.1_X)*K_C(f) &= (\alpha.1_X)*(C(f\to\bot)\to\bot) = ((\alpha.1_X\to\bot)\circledast C(f\to\bot))\to\bot\\ &\leq C((\alpha.1_X\to\bot)\circledast(f\to\bot))\to\bot = C(((\alpha.1_X)*f)\to\bot)\to\bot = K_C((\alpha.1_X)*f),\end{aligned}$$

$$\begin{aligned}(K6)\ \alpha.1_X\vee K_C(f) &= \alpha.1_X\vee(C(f\to\bot)\to\bot) = ((\alpha.1_X\to\bot)\to\bot)\vee(C(f\to\bot)\to\bot)\\ &= ((\alpha.1_X\to\bot)\wedge(C(f\to\bot)))\to\bot = C(((\alpha.1_X)\to\bot)\wedge(f\to\bot))\to\bot\\ &= C(((\alpha.1_X)\vee f)\to\bot)\to\bot = K_C(\alpha.1_X\vee f).\end{aligned}$$

(C) One can easily deduce that $K_{C_K}=K$ and $C_{K_C}=C$.

4. Characterizations of multi fuzzifying topology

In this section L is assumed to be a completely distributive complete residuated lattice, where L satisfies the double negation law.

In (Corollary 2.15 [Höhle (5)] (see Theorem 1.4.4)) proved that the L-fuzzy contiguity relations and L-fuzzifying topologies are equivalent notions if L is a completely distributive complete MV-algebra. In the following we prove that L-fuzzy contiguity relations and multi fuzzifying topology are equivalent notions just if L is a completely distributive complete residuated lattice satisfies the double negation law so that we give a generalization of U. Höhle's result.

In [Höhle (5)] from Theorems 1.4.2, 14.3, the concepts of L-fuzzifying topology and L-fuzzifying neighborhood system are equivalent notions. Then our generalization of U. Höhle's result is obtained if we prove that multi fuzzifying contiguity relation and L-fuzzifying neighborhood system are equivalent notions.

Theorem 4.1. Let N_x be an L-fuzzifying neighborhood system. Define $c_{(\varphi_x)} : X \times P\ ([X]^m) \to L$ as : $c_{(N_x)}(x,A) = N_x(X-A) \to \bot$. Then $c_{(N_x)}$ is an L-fuzzy contiguity relation induces by L-fuzzifying neighborhood system N_x. Let c be an L-fuzzy contiguity relation. Define $(N_x)_c : P\ ([X]^m) \to L$ as: $(N_x)_c(A) = c(x, X-A) \to \bot$. Then $(N_x)_c$ is an L-fuzzifying neighborhood system induces by L-fuzzy contiguity relation c. Furthermore $c_{(N_x)_c} = c$ and $(N_x)_{c_{(\varphi_x)}} = (N_x)$.

Proof. (A) (c_1) $c_{(N_x)}(x,\phi) = N_x(X-\phi) \to \bot = \top \to \bot = \bot$

$$\begin{aligned}(c_2)\ c_{(N_x)}(x, A\cup B) &= N_x(X-(A\cup B)) \to \bot = N_x((X-A)\cap(X-B)) \to \bot \\ &= ((N_x(X-A)) \wedge (N_x(X-B))) \to \bot = (N_x(X-A) \to \bot) \vee (N_x(X-B) \to \bot) \\ &= c(x,A) \vee c(x,A),\end{aligned}$$

(c_3) Let $x \in A$. Then $c_{(N_x)}(x,A) = N_x(X-A) \to \bot = \bot \to \bot = \top$

$$\begin{aligned}(c_4)\ c_{(N_x)}(x,A) &= N_x(X-A) \to \bot \geq (\bigvee_{y\notin B} (N_y(X-A) \vee N_x(B))) \to \bot \\ &= \textstyle\bigwedge_{y\in B}(N_y(X-A) \to \bot) \wedge (N_x(B^c) \to \bot) = (\bigwedge_{y\in B} c(y,A)) \wedge c(x,B).\end{aligned}$$

(B) (f_1) $(N_x)_c(X) = c(x,\phi) \to \bot = \bot \to \bot = \top$,

$$\begin{aligned}(f_2)\ (N_x)_c(A\cap B) &= c(x,(X-A)\cup(X-B)) \to \bot = (c(x,X-A) \vee c(x,X-B)) \to \bot \\ &= (c(x,X-A) \to \bot) \wedge (c(x,X-B) \to \bot) = \varphi_x(A) \wedge \varphi_x(B), \\ &= N_x(A) \wedge N_x(B),\end{aligned}$$

(u_3) Let $x \notin A$. Then $(N_x)_c(A) = c(x,X-A) \to \bot = \top \to \bot = \bot$,

$$\begin{aligned}(u_4)\ (\varphi_x)_c(A) &= c(x,X-A) \to \bot \leq ((\textstyle\bigwedge_{y\in B^c} c(y,X-A)) \wedge c(x,X-B)) \to \bot \\ &= \textstyle\bigvee_{y\in B^c}(c(y,X-A) \to \bot) \vee (c(x,X-B) \to \bot) \\ &= \textstyle\bigvee_{y\notin B}((\varphi_y)_c(A) \vee (\varphi_x)_c(B).\end{aligned}$$

$$\begin{aligned}(C)\ (\varphi_x)_{c_{(\varphi_x)}}(A) &= c_{(\varphi_x)}(x,X-A) \to \bot = (\varphi_x(A) \to \bot) \to \bot = \varphi_x(A) \quad \text{and} \\ c_{(\varphi_x)_c}(x,A) &= (\varphi_x)_c(X-A) \to \bot = (c(x,A) \to \bot) \to \bot \\ &= c(x,A).\end{aligned}$$

Definition 4.1. Let X be a nonempty set. An element $d \in L^{X\times P\ ([X]^m)}$ is called an L-fuzzy co-contiguity relation on X iff d satisfies the following axioms,

$(co-c_1)$ $d(x,X)=\top \ \ \forall\, x\in X$,

$(co-c_2)$ $d(x,A\cap B)=d(x,A)\wedge d(x,B)$, (Distributivity)

$(co-c_3)$ $d(x,A)=\bot$ whenever $x\notin A$,

$(co-c_4)$ $\bigvee_{y\notin B} d(y,A)\vee d(x,B)\geq d(x,A)$.

Theorem 4.2. Let c be an L-fuzzy contiguity relation on X. Define $d_c : X\times P(X)\rightarrow L$ as: $d_c(x,A)=c(x,X-A)\rightarrow\bot$. Then d_c is an L-fuzzy co-contiguity relation on X induces by an L-fuzzy contiguity relation c on X. Let d be an L-fuzzy co-contiguity relation on X. Define $c_d : X\times P(X)\rightarrow L$ as: $c_d(x,A)=d(x,X-A)\rightarrow\bot$. Then c_d is an L-fuzzy contiguity relation on X induces by L-fuzzy co-contiguity relation d on X. Furthermore $c_{d_c}=c$ and $d_{c_d}=d$.

Proof. (A) $(co-c_1)$ $d_c(x,X)=c(x,\phi)\rightarrow\bot=\bot\rightarrow\bot=\top$,

$$\begin{aligned}(co-c_2)d_c(x,A\cap B) &= c(x,(X-A)\cup(X-B))\rightarrow\bot=(c(x,X-A)\vee c(x,X-B))\rightarrow\bot\\ &= (c(x,X-A)\rightarrow\bot)\wedge(c(x,X-B)\rightarrow\bot)=d_c(x,A)\wedge d_c(x,B),\end{aligned}$$

$(co-c_3)$ Let $x\in X-A$. Then $c(x,X-A)=\top$. So, $d_c(x,A)=\top\rightarrow\bot=\bot$,

$$\begin{aligned}(co-c_4)d_c(x,A) &= c(x,X-A)\rightarrow\bot\leq((\textstyle\bigwedge_{y\in B^c}c(y,X-A))\wedge c(x,X-B))\rightarrow\bot\\ &\leq \textstyle\bigvee_{y\notin B}(c(y,X-A)\rightarrow\bot)\vee(c(x,X-B)\rightarrow\bot)=\bigvee_{y\notin B}d_c(y,A)\vee d_c(x,B).\end{aligned}$$

(B) (c_1) $c_d(x,\phi)=d(x,X)\rightarrow\bot=\top\rightarrow\bot=\bot$,

$$\begin{aligned}(c_2)\ c_d(x,A\cup B) &= d(x,(X-A)\cap(X-B))\rightarrow\bot=(d(x,X-A)\wedge d(x,X-B))\rightarrow\bot\\ &= (d(x,X-A)\rightarrow\bot)\vee(d(x,X-B)\rightarrow\bot)=c_d(x,A)\vee c_d(x,B),\end{aligned}$$

(c_3) Let $x\in A$. Then $c_d(x,A)=d(x,X-A)\rightarrow\bot=\bot\rightarrow\bot=\top$,

$$\begin{aligned}(c_4)\ c_d(x,A) &= d(x,X-A)\rightarrow\bot\geq(\textstyle\bigvee_{y\notin B^c}d(y,X-A))\vee d(x,X-B))\rightarrow\bot\\ &= \textstyle\bigwedge_{y\notin B^c}d(y,X-A)\rightarrow\bot)\wedge(d(x,X-B)\rightarrow\bot)\\ &= \textstyle\bigwedge_{y\in B}c_d(y,A)\wedge c_d(x,B)\end{aligned}$$

$$\begin{aligned}\text{(C) } d_{c_d}(x,A) &= c_d(x,X-A)\rightarrow\bot=(d(x,A)\rightarrow\bot)\rightarrow\bot=d(x,A) \text{ and}\\ c_{d_c}(x,A) &= d_c(x,A^c)\rightarrow\bot=(c(x,A)\rightarrow\bot)\rightarrow\bot\\ &= c(x,A).\end{aligned}$$

Theorem 4.3. The concepts of multi fuzzifying topology and L-fuzzifying co-topology are equivalent notions.

Definition 4.2. Let X be a nonempty set and Let $x \in X$. The L-fuzzifying co-neighborhood system of x is denoted by $\psi_x \in L^{P\ ([X]^m)}$ and satisfies the following conditions:

$(co-f_1)$ $\psi_x(\phi) = \bot,\ \forall x \in X,$ (Boundary conditions)

$(co-f_2)$ $\psi_x(A \cup B) = \psi_x(A) \vee \psi_x(B),$ (Intersection property)

$(co-u_3)$ $\psi_x(A) = \top$ whenever $x \in A$

$(co-u_4)$ $\psi_x(A) \geq \bigwedge_{y \in B}(\psi_y(A) \wedge \psi_x(B))\ \forall B \in P\ ([X]^m)$.

Theorem 4.4. Let φ_x be an L-fuzzifying neighborhood system. Define $(\psi_x)_{(N_x)} : P\ ([X]^m) \to L$ as: $(\psi_x)_{(N_x)}(A) = N_x(A \to \bot) \to \bot$. Then $(\psi_x)_{(N_x)}$ is an L-fuzzifying co-neighborhood system induces by L-fuzzifying neighborhood system N_x on X. Let ψ_x be an L-fuzzifying co-neighborhood system. Define $(\varphi_x)_{(\psi_x)} : P\ ([X]^m) \to L$ as:
$(N_x)_{(\psi_x)}(A) = \psi_x(A \to \bot) \to \bot$. Then $(N_x)_{(\psi_x)}$ is an L-fuzzifying neighborhood system induces by L-fuzzifying co-neighborhood system ψ_x on X. Furthermore $(\psi_x)_{(N_x)_{(\psi_x)}} = \psi_x$ and $(N_x)_{(\psi_x)_{(N_x)}} = N_x$.

Proof. (A) $(co-f_1)$ $(\psi_x)_{(N_x)}(\phi) = N_x(X) \to \bot = \top \to \bot = \bot$.

$$\begin{aligned}(co-f_2)(\psi_x)_{(N_x)}(A \cup B) \to \bot &= N_x((A \cup B) \to \bot) \to \bot = N_x((A \to \bot) \wedge (B \to \bot)) \to \bot \\ &= (N_x(A \to \bot) \to \bot) \vee (N_x(B \to \bot) \to \bot) = (\psi_x)_{(N_x)}(A) \vee (\psi_x)_{(N_x)}(B).\end{aligned}$$

$(co-u_3)$ Let $x \notin A^c$. Then $N_x(A^c) = \bot$ so that $(\psi_x)_{(N_x)}(A) = N_x(A \to \bot) \to \bot = \bot \to \bot = \top$.

$$\begin{aligned}(co-u_4)\ (\psi_x)_{(N_x)}(A) &= N_x(A \to \bot) \to \bot \geq \bigvee_{y \notin B^c}(N_y(A \to \bot) \vee N_x(B^c)) \to \bot \\ &= \bigwedge_{y \notin B^c}(N_y(A \to \bot) \to \bot) \wedge (N_x(B^c) \to \bot) \\ &= \bigwedge_{y \in B}((\psi_y)_{(N_y)}(A) \wedge ((\psi_x)_{(N_x)}(B)).\end{aligned}$$

(B) (f_1) $(N_x)_{(\psi_x)}(X) = \psi_x(\phi) \to \bot = \bot \to \bot = \top$.

$$\begin{aligned}(f_2)\ (N_x)_{(\psi_x)}(A \cap B) &= \psi_x((A \cap B) \to \bot) \to \bot = (\psi_x((A \to \bot) \cup (B \to \bot))) \to \bot \\ &= (\psi_x(A \to \bot) \vee (\psi_x(B \to \bot)) \to \bot \\ &= (\psi_x(A \to \bot) \to \bot) \wedge (\psi_x(B \to \bot) \to \bot) = (N_x)_{(\psi_x)}(A) \wedge (N_x)_{(\psi_x)}(B).\end{aligned}$$

(u_3) Let $x \notin A$. Then $(N_x)_{\psi_x}(A) = \psi_x(A \to \bot) \to \bot = \top \to \bot = \bot$.

$$\begin{aligned}(u_4)\ (N_x)_{(\psi_x)}(A) &= \psi_x(A \to \bot) \to \bot \leq (\bigwedge_{y \in B^c} \psi_y(A \to \bot) \wedge \psi_x(X - B)) \to \bot \\ &= \bigvee_{y \in B^c}(\psi_y(A \to \bot) \to \bot) \vee (\psi_x(X - B) \to \bot) \\ &= (\bigvee_{y \notin B}(N_y)_{(\psi_y)}(A)) \vee (N_x)_{(\psi_x)}(B).\end{aligned}$$

(C) $(\psi_x)_{(N_x)_{(\psi_x)}}(A) \;=\; {}_{(N_x)_{(\psi_x)}}(A \to \bot) \to \bot = (\psi_x(A) \to \bot) \to \bot = \psi_x(A)$ and

$(N_x)_{(\psi_x)_{(N_x)}}(A) \;=\; (\psi_x)_{(N_x)}(A \to \bot) \to \bot = (N_x(A) \to \bot) \to \bot = N_x(A).$

Definition 4.3. Let X be a nonempty set. A map $(\ )^\circ\colon\ P\ ([X]^m) \to L^X$ is called an L-fuzzifying interior operator if $(\ )^\circ$ satisfies the following conditions:

(1^0) $(X)^\circ = 1_X$,

(2^0) $(A \cap B)^\circ = (A)^\circ \wedge (B)^\circ$,

(3^0) $(A)^\circ \leq A$,

(4^0) $(A)^\circ(x) \leq \bigvee_{y \notin B} ((A)^\circ(y) \vee (B)^\circ(x))$.

Theorem 4.5. Let N_x be an L-fuzzifying neighborhood system. Define $(\ )^\circ\colon\ P\ ([X]^m) \to L^X$ as: $(A)^\circ_N(x) = N_x(A)$. Then $(\ )^\circ_N$ is an L-fuzzifying interior operator induces by L-fuzzifying neighborhood system N_x on X. Let $(\ )^\circ$ be an L-fuzzifying interior operator. Define $N_x : P\,([X]^m) \to L$ as: $(N_x)_{(\ )^\circ}(A) = (A)^\circ(x)$. Then $(N_x)_{(\ )^\circ}$ is an L-fuzzifying neighborhood system induces by L-fuzzifying interior operator $(\ )^\circ$ on X. Moreover $(N_x)_{(\ )^\circ_{N_x}} = N_x$ and $(\ )^\circ_{(N_x)_{(\ )^\circ}} = (\ )^\circ$.

Proof.
(A) (1°) $(X)^\circ_{N_x}(x) = N_x(X) = \top$, $\forall x \in X$. So, $(X)^\circ_{N_x} = 1_X$.

(2°) $(A \cap B)^\circ_{\varphi_x} \;=\; N_x(A \cap B) = N_x(A) \wedge N_x(A) = (A)^\circ_{N_x} \wedge (B)^\circ_{N_x}$,

(3°) $(A)^\circ_{N_x}(x) \;=\; N_x(A) \leq A(x)$,

(4°) $(A)^\circ_{N_x}(x) \;=\; N_x(A) \leq \bigvee_{y \notin B}(N_y(A) \vee N_x(B)) = \bigvee_{y \notin B}((A)^\circ_{N_x}(y) \vee (B)^\circ_{N_x}(x))$.

(B) $(f1)$ $(N_x)_{(\ )^\circ}(X) = (X)^\circ(x) = 1_X(x) = \top$, $\forall x \in X$.

$(f2)$ $(N_x)_{(\ )^\circ}(A \cap B) \;=\; (A \cap B)^\circ(x) = (A)^\circ(x) \wedge (B)^\circ(x) = (N_x)_{(\ )^\circ}(A) \wedge (N_x)_{(\ )^\circ}(B)$,

$(u3)$ Let $x \notin A$. Then, $(N_x)_{(\ )^\circ}(A) = (A)^\circ(x) \leq A(x) = \bot$. So, $(\varphi_x)_{(\ )^\circ}(A) = \bot$.

$(u4)$ $(N_x)_{(\ )^\circ}(A) \;=\; (A)^\circ(x) \leq \bigvee_{y \notin B}((A)^\circ(y) \vee (B)^\circ(x)) = \bigvee_{y \notin B}((N_y)_{(\ )^\circ}(A) \vee (N_x)_{(\ )^\circ}(B))$.

(C) $(N_x)_{(\ )^\circ_{(N_x)}}(A) = (A)^\circ_{(N_x)}(x) = N_x(A)$ and $(\ )^\circ_{(N_x)_{(\ )^\circ}}(A)(x) = (N_x)_{(\ )^\circ}(A) = (A)^\circ(x)$.

Definition 4.4. Let X be a nonempty set. A map $(\ )^-\colon\ P\ ([X]^m) \to L^X$ is called an L-fuzzifying closure operator if $(\ )^-$ satisfies the following conditions:

(1^-) $(\phi)^- = 1_\phi$,

(2^-) $(A \cup B)^- = (A)^- \vee (B)^-$, (3$^-$) $A \leq (A)^-$,

(4$^-$) $(A)^-(x) \leq \bigwedge_{y\notin B}((A)^-(y) \wedge (B)^-(x))$.

Theorem 4.6. Let ψ_x be an L-fuzzifying co-neighborhood system. Define $(\ )^-:\ P\ ([X]^m) \to L^X$ as: $(A)^-_\psi(x) = \psi_x(A)$. Then $(\ )^-_\psi$ is an L-fuzzifying closure operator induces by L-fuzzifying co-neighborhood system ψ_x on X. Let $(\ )^-$ be an L-fuzzifying closure operator. Define $\psi_x : P\ ([X]^m) \to L$ as: $(\psi_x)_{(\ )^-}(A) = A^-(x)$. Then $(\psi_x)_{(\ )^\circ}$ is an L-fuzzifying co-neighborhood system induces by L-fuzzifying closure operator $(\ )^-$ on X. Moreover $(\psi_x)_{(\ )^-_{(\psi_x)}} = \psi_x$ and $(\ )^-_{\psi_{x_{(\ )^-}}} = (\ )^-$.

Proof. (A) $(1)^-$ $(\phi)^-_\psi(x) = \psi_x(\phi) = \perp,\ \forall x \in X$. So, $(\phi)^-_\psi(x) = 1_\phi$,

$$(2^-)\ (A \cup B)^-_\psi(x) = \psi_x(A \cup B) = \psi_x(A) \vee \psi_x(B) = (A)^-_\psi(x) \vee (B)^-_\psi(x),$$

$$(3^-)\ (A)^-_\psi(x) = \psi_x(A) \geq A(x),$$

$$(4^-)\ (A)^-_\psi(x) = \psi_x(A) \geq \bigwedge_{y\notin B}(\psi_y(A) \wedge \psi_x(B)) = \bigwedge_{y\notin B}((A)^-_\psi(y) \wedge (B)^-_\psi(x)).$$

(B) $(co-f1)$ $(\psi_x)_{(\ )^-}(\phi) = \phi^-(x) = 1_\phi,\ \forall x \in X$. So, $(\psi_x)_{(\ )^-}(\phi) = \perp$.

$$(co-f2)\ (\psi_x)_{(\ )^-}(A \cup B) = (A \cup B)^-(x) = (A)^-(x) \vee (B)^-(x) = (\psi_x)_{(\ )^-}(A) \vee (\psi_x)_{(\ )^-}(B),$$

$(co-u3)$ Let $x \in A$. Then $(\psi_x)_{(\ )^-}(A) = (A)^-(x) = \top$. So, $(\psi_x)_{(\ )^-}(A) = \top$,

$$(co-u4)\ (\psi_x)_{(\ )^-}(A) = (A)^-(x) \leq \bigwedge_{y\notin B}((A)^-(y) \wedge (B)^-(x)) = \bigwedge_{y\notin B}(((A) \vee (\psi_x)_{(\ )^{\psi_y)_{(\ )^-}}-}(B)).$$

(C) $(\psi_x)_{(\ )^-_{(\psi_x)}}(A) = (A)^-_\psi(x) = \psi_x(A)$ and $(\ )^-_{\psi_{x_{(\ )^-}}}(A) = (\psi_x)_{(\ )^-}(A) = (A)^-(x)$.

Remark 4.1. From (Corollary 2.15 [Höhle (1999)] (see Theorem 1.1)) and Theorems 3.2.1, 3.2.2, 4.1, 4.2,4.3, 4.4, 4.5, 4.6 one can have the following Theorem:

Theorem 4.7. If L is a completely distributive complete MV-algebra then the concepts of stratified and transitive L-topologies, stratified and transitive co-L-topologies, stratified and transitive L-interior operators, stratified and transitive L-closure operators, L-fuzzy contiguity relations, L-fuzzifying neighborhood systems, L-fuzzy co-contiguity relations, multi fuzzifying topologies, L-fuzzifying co-topologies, L-fuzzifying co-neighborhood systems, L-fuzzifying interior operators, L-fuzzifying closure operators are equivalent notions.

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