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Some four-parameter trigonometric generalizations of the Hilbert integral inequality

Christophe Chesneau*
Department of Mathematics, LMNO, University of Caen-Normandie,
Caen, France. ORCID iD: 0000-0002-1522-9292

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Abstract: In this paper, we propose several trigonometric generalizations of the Hilbert integral inequality. They are characterized by the inclusion of four adjustable parameters and mainly cosine and sine (or sinc) functions, which remain an under-explored area in the current literature.

Key words: Hilbert integral inequality, trigonometric functions, sine cardinal function, trigonometric inequalities.

1. Introduction

Integral inequalities are one of the most basic tools of mathematical analysis. Essentially, they provide bounds for integrals of functions under various conditions. In particular, they facilitate the derivation of integral approximations and the establishment of convergence criteria. This makes them crucial in both theoretical and applied mathematics. See [1] and [10]. Among the famous integral inequalities, the classical Hilbert integral inequality (HII) stands out as one of the most important. It is particularly well known for its ability to bound integrals involving products of functions. The (classical) HII can be expressed formally as follows: For any square-integrable functions $p : [0, +\infty) \to [0, +\infty)$ and $q : [0, +\infty) \to [0, +\infty)$, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{p(x)q(y)}{x+y} dxdy < \pi ||p||_2 ||q||_2, \tag{1}$$

where, for any square-integrable function $r : [0, +\infty) \to [0, +\infty)$, the corresponding L_2 norm is defined by

$$||r||_2 = \left\{ \int_0^{+\infty} [r(z)]^2 dz \right\}^{1/2}.$$

Since its discovery, the HII has inspired a great deal of research. See, for example, [6], [9], [8], [7], [12], [13], and [2]. It has also been generalized in various ways. A comprehensive state of the art can be found in [3] and [14], together with the references therein.

Surprisingly, if we focus on upper bounds of the exact form "constant multiplied by $||p||_2||q||_2$", trigonometric types of the HII are relatively rare. The most famous of them are defined using the arctangent function, mainly because of its integrability property and its singular formula $\arctan(z) + \arctan(1/z) = \pi/2$ for $z > 0$. For example, the following is proved in [12]:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{\arctan(x/y)}{\max(x, y)} p(x)q(y) dxdy < \pi ||p||_2 ||q||_2.$$

*Correspondence: christophe.chesneau@gmail.com

We may also mention the one in [13], which can be formulated as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\arctan\left[(x/y)^{1/2}\right]}{x+y}p(x)q(y)dxdy<\frac{\pi^2}{4}||p||_2||q||_2.$$

Another notable one-parameter trigonometric type of HII is proposed in [14, Example 2.1.9]. It is formulated as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\left\{\arctan[(x/y)^{1/2}]\right\}^{\alpha}}{x+y}p(x)q(y)dxdy<\frac{\pi}{\alpha+1}\left(\frac{\pi}{2}\right)^{\alpha}||p||_2||q||_2,$$

for any $\alpha>-1$. With $\alpha=1$, we get the above inequality, and with $\alpha=0$, we get the classical HII.

A more recent inequality of this type is also given in [4]. It can be written as

$$\int_0^{+\infty}\int_0^{+\infty}\frac{\arctan(x/y)}{x+y}p(x)q(y)dxdy\leq\frac{\pi^2}{4}||p||_2||q||_2.$$

To the best of our knowledge, variants of the HII based on other trigonometric functions with upper bounds of the exact form "constant multiplied by $||p||_2||q||_2$" are almost non-existent in the literature. With this in mind, in this paper, we innovate by investigating two new generalizations of the HII centered on the cosine and sine functions. In the first main result, we determine a sharp and tractable constant $C>0$ such that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy\leq C||p||_2||q||_2,$$

where α, β, γ and θ are four adjustable parameters. So α activates the main trigonometric term, i.e., $\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]$, β and θ can be thought of as angle parameters, and γ modulates the variable x in the denominator term. We want to have as few constraints as possible on these parameters. The constant C is logically dependent on these parameters. In our second main result, under a similar parameter setting, we determine a sharp and manageable constant $D>0$ such that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-(\alpha/\beta)(y/x)^{1/2}\sin\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy\leq D||p||_2||q||_2,$$

which can be reformulated as

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\,\mathrm{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy\leq D||p||_2||q||_2,$$

where we have considered the sinc function defined by

$$\mathrm{sinc}(z)=\begin{cases}\dfrac{\sin(z)}{z} & \text{if } z\neq 0\\ 1 & \text{if } z=0\end{cases}. \tag{2}$$

See [11]. Again, the parameter α activates the main trigonometric term, here $\mathrm{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]$, β and θ can be thought of as angle parameters, and γ modulates the variable x in the denominator term. The constant D logically depends on the parameters involved. For both results, the classical HII is recovered by taking $\alpha = 0$ and $\gamma = 1$, which explains the trigonometric nature of the generalizations. Based on these results, several special cases for certain values of the parameters are discussed. To the best of our knowledge, they are new in the literature and offer a trigonometric perspective for more types of integral inequalities related to the HII.

The rest of the paper consists of Section 2, which focuses on the first main result along with some special cases, and Section 3, which details the second main result. A conclusion is given in Section 4.

2. Cosine type HII

The proposal for a four-parameter cosine type of HII is given below.

Theorem 2.1. *Let $\alpha \in [-1, 1]$, $\beta \geq 0$, $\gamma > 0$ and $\theta \geq 0$. For any square-integrable functions $p : [0, +\infty) \to [0, +\infty)$ and $q : [0, +\infty) \to [0, +\infty)$, we have*

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1 - \alpha \cos\left[\beta(x/y)^{1/2}\right] \cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x)q(y)dxdy \leq C_{\alpha,\beta,\gamma,\theta}||p||_2||q||_2, \tag{3}$$

where

$$C_{\alpha,\beta,\gamma,\theta} = \frac{\pi}{\gamma^{1/2}} \left\{1 - \frac{\alpha}{2} \left[e^{-|\beta-\theta|/\gamma^{1/2}} + e^{-(\beta+\theta)/\gamma^{1/2}}\right]\right\}. \tag{4}$$

Proof. First of all, let us note that $\cos(z) \in [-1, 1]$ for any $z \in \mathbb{R}$. Therefore, for any $x > 0$ and $y > 0$, since $\alpha \in [-1, 1]$, we have

$$1 - \alpha \cos\left[\beta\left(\frac{x}{y}\right)^{1/2}\right] \cos\left[\theta\left(\frac{x}{y}\right)^{1/2}\right] \geq 1 - |\alpha| \left|\cos\left[\beta\left(\frac{x}{y}\right)^{1/2}\right]\right| \left|\cos\left[\theta\left(\frac{x}{y}\right)^{1/2}\right]\right|$$
$$\geq 1 - |\alpha| \geq 0.$$

As a result, since $\gamma > 0$, we have

$$\frac{1 - \alpha \cos\left[\beta(x/y)^{1/2}\right] \cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} \geq 0,$$

which allows it to be raised to any positive exponent. It follows from this, the Cauchy-Schwarz inequality (or

Hölder inequality of exponent 2) with respect to the variables x and y, and the Fubini-Tonelli theorem that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\left\{\left[\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}\right]^{1/2}\left(\frac{x}{y}\right)^{1/4}p(x)\right\}\\
&\times\left\{\left[\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}\right]^{1/2}\left(\frac{y}{x}\right)^{1/4}q(y)\right\}dxdy\\
&\leq\left\{\int_0^{+\infty}\left[\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}\left(\frac{x}{y}\right)^{1/2}dy\right][p(x)]^2dx\right\}^{1/2}\\
&\times\left\{\int_0^{+\infty}\left[\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}\left(\frac{y}{x}\right)^{1/2}dx\right][q(y)]^2dy\right\}^{1/2}\\
&=\left\{\int_0^{+\infty}S(x)[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}T(y)[q(y)]^2dy\right\}^{1/2},
\end{aligned}\tag{5}$$

where

$$S(x)=\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}\left(\frac{x}{y}\right)^{1/2}dy$$

and

$$T(y)=\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}\left(\frac{y}{x}\right)^{1/2}dx.$$

Let us now examine the expressions of $S(x)$ and $T(y)$. To do this, the known lemma below is crucial.

Lemma 2.1. *For any $a\geq 0$, $b\geq 0$ and $c>0$, we have*

$$\int_0^{+\infty}\frac{\cos(ax)\cos(bx)}{c+x^2}dx=\frac{\pi}{4c^{1/2}}\left[e^{-|a-b|c^{1/2}}+e^{-(a+b)c^{1/2}}\right].$$

We refer to [5, Formula number 3.742.3].

By considering the change of variables $y=x/u^2$, and Lemma 2.1 with appropriate choices for the parameters, we obtain

$$\begin{aligned}
S(x) &= \int_{+\infty}^{0} \frac{1-\alpha\cos(\beta u)\cos(\theta u)}{\gamma x + x/u^2}(u^2)^{1/2}\left(-\frac{2x}{u^3}\right)du \\
&= 2\int_0^{+\infty} \frac{1-\alpha\cos(\beta u)\cos(\theta u)}{1+\gamma u^2}du \\
&= \frac{2}{\gamma}\left[\int_0^{+\infty}\frac{1}{1/\gamma+u^2}du - \alpha\int_0^{+\infty}\frac{\cos(\beta u)\cos(\theta u)}{1/\gamma+u^2}du\right] \\
&= \frac{2}{\gamma}\left\{\frac{\pi\gamma^{1/2}}{2} - \alpha\frac{\pi\gamma^{1/2}}{4}\left[e^{-|\beta-\theta|/\gamma^{1/2}} + e^{-(\beta+\theta)/\gamma^{1/2}}\right]\right\} \\
&= \frac{\pi}{\gamma^{1/2}}\left\{1-\frac{\alpha}{2}\left[e^{-|\beta-\theta|/\gamma^{1/2}} + e^{-(\beta+\theta)/\gamma^{1/2}}\right]\right\} \\
&= C_{\alpha,\beta,\gamma,\theta}.
\end{aligned}$$

Let us now concentrate on $T(y)$. Considering the change of the variables $x = yu^2$ and the second step of the previous development, we get more directly

$$\begin{aligned}
T(y) &= \int_0^{+\infty} \frac{1-\alpha\cos(\beta u)\cos(\theta u)}{\gamma y u^2 + y}\left(\frac{1}{u^2}\right)^{1/2}(2yu)du \\
&= 2\int_0^{+\infty} \frac{1-\alpha\cos(\beta u)\cos(\theta u)}{1+\gamma u^2}du \\
&= C_{\alpha,\beta,\gamma,\theta}.
\end{aligned}$$

Hence, both $S(x)$ and $T(y)$ are equal to $C_{\alpha,\beta,\gamma,\theta}$. Owing to Equation (5), we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y}p(x)q(y)dxdy \\
&\leq \left\{\int_0^{+\infty} C_{\alpha,\beta,\gamma,\theta}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty} C_{\alpha,\beta,\gamma,\theta}[q(y)]^2dy\right\}^{1/2} \\
&= C_{\alpha,\beta,\gamma,\theta}||p||_2||q||_2.
\end{aligned}$$

The result is proved. □

We can derive the following inequalities from Theorem 2.1:

- By choosing $\alpha = 0$ (which immediately disables the parameters β and θ) and $\gamma = 1$, we find that

$$C_{0,\beta,1,\theta} = \frac{\pi}{1^{1/2}}(1-0) = \pi.$$

  In this case, Equation (3) is reduced to the classical HII, i.e., with the expected constant π.

- By choosing $\theta = 0$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]}{\gamma x + y}p(x)q(y)dxdy \leq C_{\alpha,\beta,\gamma,0}||p||_2||q||_2,$$

where

$$C_{\alpha,\beta,\gamma,0} = \frac{\pi}{\gamma^{1/2}}\left[1-\alpha e^{-\beta/\gamma^{1/2}}\right]. \tag{6}$$

Two special cases of this formula are now highlighted. If we take $\beta = 2\nu$ with $\nu \geq 0$ and $\alpha = 1$, and use the formula $[1-\cos(2z)]/2 = [\sin(z)]^2$ for any $z \in \mathbb{R}$, we get

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\left\{\sin\left[\nu(x/y)^{1/2}\right]\right\}^2}{\gamma x + y} p(x)q(y)dxdy \leq C^{\dagger}||p||_2||q||_2,$$

where

$$C^{\dagger} = \frac{1}{2}C_{1,2\nu,\gamma,0} = \frac{\pi}{2\gamma^{1/2}}\left[1-e^{-2\nu/\gamma^{1/2}}\right].$$

Similarly, if we take $\beta = 2\nu$ with $\nu \geq 0$ and $\alpha = -1$, and use the formula $[1+\cos(2z)]/2 = [\cos(z)]^2$ for any $z \in \mathbb{R}$, we get

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\left\{\cos\left[\nu(x/y)^{1/2}\right]\right\}^2}{\gamma x + y} p(x)q(y)dxdy \leq C^{\ddagger}||p||_2||q||_2,$$

where

$$C^{\ddagger} = \frac{1}{2}C_{-1,2\nu,\gamma,0} = \frac{\pi}{2\gamma^{1/2}}\left[1+e^{-2\nu/\gamma^{1/2}}\right].$$

- By choosing $\beta = \theta$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1-\alpha\left\{\cos\left[\beta(x/y)^{1/2}\right]\right\}^2}{\gamma x + y} p(x)q(y)dxdy \leq C_{\alpha,\beta,\gamma,\beta}||p||_2||q||_2,$$

where

$$C_{\alpha,\beta,\gamma,\beta} = \frac{\pi}{\gamma^{1/2}}\left\{1-\frac{\alpha}{2}\left[1+e^{-2\beta/\gamma^{1/2}}\right]\right\}.$$

By using the classical trigonometric formula $[\cos(z)]^2+[\sin(z)]^2 = 1$ for any $z \in \mathbb{R}$, we get $\left\{\cos\left[\beta(x/y)^{1/2}\right]\right\}^2 = 1-\left\{\sin\left[\beta(x/y)^{1/2}\right]\right\}^2$, and we also establish that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1-\alpha+\alpha\left\{\sin\left[\beta(x/y)^{1/2}\right]\right\}^2}{\gamma x + y} p(x)q(y)dxdy \leq C_{\alpha,\beta,\gamma,\beta}||p||_2||q||_2.$$

In addition, from Theorem 2.1, the following type of inverse trigonometric HII can be derived:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{p(x)q(y)}{\gamma x + y} dxdy - C_{\alpha,\beta,\gamma,\theta}||p||_2||q||_2$$
$$\leq \alpha\int_0^{+\infty}\int_0^{+\infty} \frac{\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x)q(y)dxdy.$$

The facts that α can be negative or positive, and that $\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]$ is relatively complex to handle with changing signs in functions of β, θ, x and y, make this inequality somewhat sophisticated and of potential interest. To the best of our knowledge, all the trigonometric (cosine) types of HII presented here are new to the literature. Of course, other examples can be derived by choosing different values for the parameters involved.

Another trigonometric integral result related to Theorem 2.1 is presented below.

Theorem 2.2. *Let $\alpha \in [-1,1]$, $\beta \geq 0$, $\gamma > 0$ and $\theta \geq 0$. For any square-integrable function $p : [0,+\infty) \to [0,+\infty)$, we have*

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)dx\right]^2 dy \leq C^2_{\alpha,\beta,\gamma,\theta}||p||_2^2,$$

where $C_{\alpha,\beta,\gamma,\theta}$ is described in Equation (4). In addition, the above inequality implies Equation (3).

Proof. We introduce the intermediary integral function $r : [0,+\infty) \to [0,+\infty)$ defined by

$$r(y) = \int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)dx.$$

By applying Theorem 2.1 with the functions $p(x)$ and $q(y) = r(y)$, we get

$$\begin{aligned}||r||_2^2 &= \int_0^{+\infty}\left[\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)dx\right]r(y)dy\\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)r(y)dxdy\\ &\leq C_{\alpha,\beta,\gamma,\theta}||p||_2||r||_2.\end{aligned}$$

By simplifying with $||r||_2$ on the both sides and raising to the square exponent, we get

$$||r||_2^2 \leq C^2_{\alpha,\beta,\gamma,\theta}||p||_2^2,$$

which is equivalent to

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)dx\right]^2 dy \leq C^2_{\alpha,\beta,\gamma,\theta}||p||_2^2.$$

This establishes the desired inequality.

For the reciprocal case, let us suppose that

$$\int_0^{+\infty}\left[\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)dx\right]^2 dy \leq C^2_{\alpha,\beta,\gamma,\theta}||p||_2^2$$

and consider an arbitrary square-integrable function $q : [0,+\infty) \to [0,+\infty)$. It follows from the Cauchy-Schwarz inequality with respect to the variable y and the above assumption that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$=\int_0^{+\infty}\left[\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)dx\right]q(y)dy$$
$$\leq\left\{\int_0^{+\infty}\left[\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)dx\right]^2dy\right\}^{1/2}||q||_2$$
$$\leq\left\{C^2_{\alpha,\beta,\gamma,\theta}||p||_2^2\right\}^{1/2}||q||_2=C_{\alpha,\beta,\gamma,\theta}||p||_2||q||_2.$$

Equation (3) is established, which completes the proof. □

Theorem 2.2 thus provides an alternative formulation of Theorem 2.1, and a different way of looking at the effects of the parameters involved and $p(x)$.

To conclude this section, we mention that Theorems 2.1 and 2.2 are also valid if we replace the main integrated term, i.e.,

$$\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y},$$

by

$$\frac{1-\alpha\cos\left[\beta(xy)^{1/2}\right]\cos\left[\theta(xy)^{1/2}\right]}{1+\gamma xy}.$$

The proofs are similar, except that the change of the variable $u^2=xy$ must be considered for the analogues of $S(x)$ and $T(y)$ in Theorem 2.1, from which Theorem 2.2 can be derived. These results may still be of interest, although they do not correspond to generalizations of the HII.

3. Sine type HII

This section follows the same structure of developments as the previous section, with another kind of trigonometric type HII. More specifically, the proposal for a four-parameter sine type HII is given below.

Theorem 3.1. *Let* $\alpha\in[-1,1]$, $\beta\geq 0$, $\gamma>0$ *and* $\theta\geq 0$. *For any square-integrable functions* $p:[0,+\infty)\to[0,+\infty)$ *and* $q:[0,+\infty)\to[0,+\infty)$, *we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\,\mathrm{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy\leq D_{\alpha,\beta,\gamma,\theta}||p||_2||q||_2,\tag{7}$$

where the definition of the sine cardinal function is recalled in Equation (2) and

$$D_{\alpha,\beta,\gamma,\theta}=\begin{cases}\pi\left[\frac{1}{\gamma^{1/2}}-\frac{\alpha}{\beta}\left\{1-e^{-\beta/\gamma^{1/2}}\cosh\left[\frac{\theta}{\gamma^{1/2}}\right]\right\}\right] & \text{if } \beta\geq\theta\\ \pi\left\{\frac{1}{\gamma^{1/2}}-\frac{\alpha}{\beta}e^{-\theta/\gamma^{1/2}}\sinh\left[\frac{\beta}{\gamma^{1/2}}\right]\right\} & \text{if } \theta\geq\beta\end{cases},\tag{8}$$

with $\cosh(z)=(e^z+e^{-z})/2$ *and* $\sinh(z)=(e^z-e^{-z})/2$ *for any* $z\in\mathbb{R}$.

Proof. It is well known that $\cos(z) \in [-1,1]$ and $\operatorname{sinc}(z) \in [-1,1]$ for any $z \in \mathbb{R}$. Therefore, for any $x > 0$ and $y > 0$, since $\alpha \in [-1,1]$, we get

$$1 - \alpha \operatorname{sinc}\left[\beta\left(\frac{x}{y}\right)^{1/2}\right]\cos\left[\theta\left(\frac{x}{y}\right)^{1/2}\right] \geq 1 - |\alpha|\left|\operatorname{sinc}\left[\beta\left(\frac{x}{y}\right)^{1/2}\right]\right|\left|\cos\left[\theta\left(\frac{x}{y}\right)^{1/2}\right]\right|$$
$$\geq 1 - |\alpha| \geq 0.$$

As a result, since $\gamma > 0$, we have

$$\frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} \geq 0.$$

We can therefore set the main integrated term to any positive exponent. It follows from this, the Cauchy-Schwarz inequality with respect to the variables x and y, and the Fubini-Tonelli theorem that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x)q(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} \left\{\left[\frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y}\right]^{1/2}\left(\frac{x}{y}\right)^{1/4} p(x)\right\}$$
$$\times \left\{\left[\frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y}\right]^{1/2}\left(\frac{y}{x}\right)^{1/4} q(y)\right\} dxdy$$
$$\leq \left\{\int_0^{+\infty} U(x)[p(x)]^2 dx\right\}^{1/2}\left\{\int_0^{+\infty} V(y)[q(y)]^2 dy\right\}^{1/2}, \tag{9}$$

where

$$U(x) = \int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y}\left(\frac{x}{y}\right)^{1/2} dy$$

and

$$V(y) = \int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y}\left(\frac{y}{x}\right)^{1/2} dx.$$

Now let us determine $U(x)$ and $V(y)$. The lemma below is central to this.

Lemma 3.1. *For any $a \geq 0$, $b \geq 0$ and $c > 0$, we have*

$$\int_0^{+\infty} \frac{\sin(ax)\cos(bx)}{x(c+x^2)} dx = \begin{cases} \dfrac{\pi}{2c}\left\{1 - e^{-ac^{1/2}}\cosh[bc^{1/2}]\right\} & \text{if } a \geq b \\ \dfrac{\pi}{2c} e^{-bc^{1/2}}\sinh[ac^{1/2}] & \text{if } b \geq a \end{cases}.$$

We refer to [5, Formula number 3.725.3].

By considering the change of variables $y = x/u^2$, we obtain

$$\begin{aligned} U(x) &= \int_{+\infty}^{0} \frac{1-\alpha\,\mathrm{sinc}(\beta u)\cos(\theta u)}{\gamma x + x/u^2}(u^2)^{1/2}\left(-\frac{2x}{u^3}\right)du \\ &= 2\int_0^{+\infty} \frac{1-\alpha\,\mathrm{sinc}(\beta u)\cos(\theta u)}{1+\gamma u^2}du \\ &= \frac{2}{\gamma}\left[\int_0^{+\infty}\frac{1}{1/\gamma+u^2}du - \alpha\int_0^{+\infty}\frac{\mathrm{sinc}(\beta u)\cos(\theta u)}{1/\gamma+u^2}du\right] \\ &= \frac{2}{\gamma}\left\{\frac{\pi\gamma^{1/2}}{2} - \frac{\alpha}{\beta}W_{\beta,\gamma,\theta}\right\}, \end{aligned}$$

where

$$W_{\beta,\gamma,\theta} = \int_0^{+\infty}\frac{\sin(\beta u)\cos(\theta u)}{u(1/\gamma+u^2)}du.$$

This integral can be determined by using Lemma 3.1 with appropriate choices for the parameters. We get

$$W_{\beta,\gamma,\theta} = \begin{cases} \dfrac{\pi\gamma}{2}\left\{1 - e^{-\beta/\gamma^{1/2}}\cosh\left[\dfrac{\theta}{\gamma^{1/2}}\right]\right\} & \text{if } \beta \geq \theta \\ \dfrac{\pi\gamma}{2}e^{-\theta/\gamma^{1/2}}\sinh\left[\dfrac{\beta}{\gamma^{1/2}}\right] & \text{if } \theta \geq \beta \end{cases}.$$

We thus obtain

$$U(x) = \begin{cases} \pi\left[\dfrac{1}{\gamma^{1/2}} - \dfrac{\alpha}{\beta}\left\{1 - e^{-\beta/\gamma^{1/2}}\cosh\left[\dfrac{\theta}{\gamma^{1/2}}\right]\right\}\right] & \text{if } \beta \geq \theta \\ \pi\left\{\dfrac{1}{\gamma^{1/2}} - \dfrac{\alpha}{\beta}e^{-\theta/\gamma^{1/2}}\sinh\left[\dfrac{\beta}{\gamma^{1/2}}\right]\right\} & \text{if } \theta \geq \beta \end{cases} = D_{\alpha,\beta,\gamma,\theta}.$$

Let us now consider $V(y)$. Taking into account the change of variables $x = yu^2$ and the previous development, we get directly

$$\begin{aligned} V(y) &= \int_0^{+\infty}\frac{1-\alpha\,\mathrm{sinc}(\beta u)\cos(\theta u)}{\gamma y u^2 + y}\left(\frac{1}{u^2}\right)^{1/2}(2yu)du \\ &= 2\int_0^{+\infty}\frac{1-\alpha\,\mathrm{sinc}(\beta u)\cos(\theta u)}{1+\gamma u^2}du \\ &= D_{\alpha,\beta,\gamma,\theta}. \end{aligned}$$

So both $U(x)$ and $V(y)$ are equal to $D_{\alpha,\beta,\gamma,\theta}$. Because of Equation (9), we get

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\,\mathrm{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y}p(x)q(y)dxdy \\ &\leq \left\{\int_0^{+\infty}D_{\alpha,\beta,\gamma,\theta}[p(x)]^2dx\right\}^{1/2}\left\{\int_0^{+\infty}D_{\alpha,\beta,\gamma,\theta}[q(y)]^2dy\right\}^{1/2} \\ &= D_{\alpha,\beta,\gamma,\theta}||p||_2||q||_2. \end{aligned}$$

The desired result is obtained. □

We can deduce the following inequalities from Theorem 3.1:

- By choosing $\alpha = 0$ (which immediately disables the parameters β and θ) and $\gamma = 1$, we find that

$$D_{0,\beta,1,\theta} = \pi \left[\frac{1}{1^{1/2}} - 0 \right] = \pi.$$

  In this case, Equation (7) is reduced to the classical HII.

- By choosing $\theta = 0$ and $\beta > 0$, we obtain

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta (x/y)^{1/2}\right]}{\gamma x + y} p(x) q(y) dx dy \leq D_{\alpha,\beta,\gamma,0} ||p||_2 ||q||_2,$$

  where

$$D_{\alpha,\beta,\gamma,0} = \pi \left[\frac{1}{\gamma^{1/2}} - \frac{\alpha}{\beta} \left\{ 1 - e^{-\beta/\gamma^{1/2}} \right\} \right].$$

  This inequality can be also expressed as

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1 - (\alpha/\beta)(y/x)^{1/2} \sin\left[\beta (x/y)^{1/2}\right]}{\gamma x + y} p(x) q(y) dx dy \leq D_{\alpha,\beta,\gamma,0} ||p||_2 ||q||_2.$$

- By choosing $\beta = 0$, we obtain

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1 - \alpha \cos\left[\theta (x/y)^{1/2}\right]}{\gamma x + y} p(x) q(y) dx dy \leq D_{\alpha,0,\gamma,\theta} ||p||_2 ||q||_2,$$

  where

$$D_{\alpha,0,\gamma,\theta} = \lim_{\beta \to 0} D_{\alpha,\beta,\gamma,\theta} = \lim_{\beta \to 0} \pi \left\{ \frac{1}{\gamma^{1/2}} - \frac{\alpha}{\beta} e^{-\theta/\gamma^{1/2}} \sinh\left[\frac{\beta}{\gamma^{1/2}} \right] \right\}$$
$$= \pi \left[\frac{1}{\gamma^{1/2}} - \frac{\alpha}{\gamma^{1/2}} e^{-\theta/\gamma^{1/2}} \right] = \frac{\pi}{\gamma^{1/2}} \left[1 - \alpha e^{-\theta/\gamma^{1/2}} \right].$$

  As expected, we get a result similar to that of Equation (6) (except that we have reversed the roles of β and θ).

In addition, from Theorem 3.1, the following type of inverse trigonometric HII can be derived:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{p(x) q(y)}{\gamma x + y} dx dy - D_{\alpha,\beta,\gamma,\theta} ||p||_2 ||q||_2$$
$$\leq \alpha \int_0^{+\infty} \int_0^{+\infty} \frac{\operatorname{sinc}\left[\beta (x/y)^{1/2}\right] \cos\left[\theta (x/y)^{1/2}\right]}{\gamma x + y} p(x) q(y) dx dy.$$

To the best of our knowledge, all the trigonometric (sinc) types of HII presented here are new to the literature. Other examples can be derived by choosing different values for the parameters involved.

Another trigonometric integral result that is related to Theorem 3.1 is presented below.

Theorem 3.2. *Let* $\alpha \in [-1,1]$, $\beta \geq 0$, $\gamma > 0$ *and* $\theta \geq 0$. *For any square-integrable function* $p : [0,+\infty) \to [0,+\infty)$, *we have*

$$\int_0^{+\infty} \left[\int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right] \cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x) dx \right]^2 dy \leq D^2_{\alpha,\beta,\gamma,\theta} ||p||_2^2,$$

where $D_{\alpha,\beta,\gamma,\theta}$ *is described in Equation* (8). *In addition, the above inequality implies Equation* (7).

Proof. We introduce the following intermediary integral function $s : [0,+\infty) \to [0,+\infty)$:

$$s(y) = \int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right] \cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x) dx.$$

It follows from Theorem 3.1 with the functions $p(x)$ and $q(y) = s(y)$ that

$$\begin{aligned} ||s||_2^2 &= \int_0^{+\infty} \left[\int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right] \cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x) dx \right] s(y) dy \\ &= \int_0^{+\infty} \int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right] \cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x) s(y) dx dy \\ &\leq D_{\alpha,\beta,\gamma,\theta} ||p||_2 ||s||_2. \end{aligned}$$

By simplification with $||s||_2$ on both sides and raising to the square exponent, we have

$$||s||_2^2 \leq D^2_{\alpha,\beta,\gamma,\theta} ||p||_2^2,$$

which reads as

$$\int_0^{+\infty} \left[\int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right] \cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x) dx \right]^2 dy \leq D^2_{\alpha,\beta,\gamma,\theta} ||p||_2^2.$$

The stated inequality is demonstrated.

For the equivalence, let us assume that

$$\int_0^{+\infty} \left[\int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right] \cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x) dx \right]^2 dy \leq D^2_{\alpha,\beta,\gamma,\theta} ||p||_2^2.$$

Let us consider a square-integrable function $q : [0,+\infty) \to [0,+\infty)$. It follows from the Cauchy-Schwarz inequality with respect to the variable y and the above assumption that

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right] \cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x) q(y) dx dy \\ &\leq \left\{ \int_0^{+\infty} \left[\int_0^{+\infty} \frac{1 - \alpha \operatorname{sinc}\left[\beta(x/y)^{1/2}\right] \cos\left[\theta(x/y)^{1/2}\right]}{\gamma x + y} p(x) dx \right]^2 dy \right\}^{1/2} ||q||_2 \\ &\leq \left\{ D^2_{\alpha,\beta,\gamma,\theta} ||p||_2^2 \right\}^{1/2} ||q||_2 = D_{\alpha,\beta,\gamma,\theta} ||p||_2 ||q||_2. \end{aligned}$$

This completes the proof by establishing Equation (7). □

Theorem 3.2 thus provides an alternative way of expressing Theorem 3.1. It also allows a deeper understanding of the implications of the parameters α, β, γ and θ.

We mention that Theorems 3.1 and 3.2 are also valid if we replace the main integrated term, i.e.,

$$\frac{1-\alpha\,\mathrm{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y},$$

by

$$\frac{1-\alpha\,\mathrm{sinc}\left[\beta(xy)^{1/2}\right]\cos\left[\theta(xy)^{1/2}\right]}{1+\gamma xy}.$$

The proofs are similar, except that the change of the variable $u^2 = xy$ must be considered for the analogues of $U(x)$ and $V(y)$ in Theorem 3.1. Theorem 3.2 can then be derived. These results are not generalizations of the HII, but may be of independent interest.

To conclude this section, we present a proposition that is a consequence of Theorems 2.1 and 3.1.

Proposition 3.1. *Let $\alpha \in [-1,1]$, $\beta \geq 0$, $\gamma > 0$, $\rho \geq 0$, $\theta \geq 0$ and $\omega \in [0,1]$. For any square-integrable functions $p : [0,+\infty) \to [0,+\infty)$ and $q : [0,+\infty) \to [0,+\infty)$, the three inequalities below hold.*

1. *We have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\left\{\omega\cos\left[\beta(x/y)^{1/2}\right]+(1-\omega)\,\mathrm{sinc}\left[\rho(x/y)^{1/2}\right]\right\}\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$\leq E_{\alpha,\beta,\gamma,\rho,\theta,\omega}||p||_2||q||_2,$$

where

$$E_{\alpha,\beta,\gamma,\rho,\theta,\omega} = \omega C_{\alpha,\beta,\gamma,\theta} + (1-\omega)D_{\alpha,\rho,\gamma,\theta},$$

$C_{\alpha,\beta,\gamma,\theta}$ is given in Equation (4) and $D_{\alpha,\rho,\gamma,\theta}$ is given in Equation (8).

2. *We have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\left\{\omega\cos\left[\beta(x/y)^{1/2}\right]+(1-\omega)\cos\left[\rho(x/y)^{1/2}\right]\right\}\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$\leq F_{\alpha,\beta,\gamma,\rho,\theta,\omega}||p||_2||q||_2,$$

where

$$F_{\alpha,\beta,\gamma,\rho,\theta,\omega} = \omega C_{\alpha,\beta,\gamma,\theta} + (1-\omega)C_{\alpha,\rho,\gamma,\theta}.$$

3. *We have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\left\{\omega\,\mathrm{sinc}\left[\beta(x/y)^{1/2}\right]+(1-\omega)\,\mathrm{sinc}\left[\rho(x/y)^{1/2}\right]\right\}\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$\leq G_{\alpha,\beta,\gamma,\rho,\theta,\omega}||p||_2||q||_2,$$

where

$$G_{\alpha,\beta,\gamma,\rho,\theta,\omega} = \omega D_{\alpha,\beta,\gamma,\theta} + (1-\omega)D_{\alpha,\rho,\gamma,\theta}.$$

Proof.

1. An analysis of the factorization reveals a linear convex combination with two components. Using this and the inequalities in Theorems 2.1 and 3.1, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\left\{\omega\cos\left[\beta(x/y)^{1/2}\right]+(1-\omega)\operatorname{sinc}\left[\rho(x/y)^{1/2}\right]\right\}\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$=\omega\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$+(1-\omega)\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\operatorname{sinc}\left[\rho(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$\leq \omega C_{\alpha,\beta,\gamma,\theta}+(1-\omega)D_{\alpha,\rho,\gamma,\theta}=E_{\alpha,\beta,\gamma,\rho,\theta,\omega}.$$

2. Similarly, but with the use of only Theorem 2.1, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\left\{\omega\cos\left[\beta(x/y)^{1/2}\right]+(1-\omega)\cos\left[\rho(x/y)^{1/2}\right]\right\}\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$=\omega\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\cos\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$+(1-\omega)\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\cos\left[\rho(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$\leq \omega C_{\alpha,\beta,\gamma,\theta}+(1-\omega)C_{\alpha,\rho,\gamma,\theta}=F_{\alpha,\beta,\gamma,\rho,\theta,\omega}.$$

3. Similarly, but with the use of only Theorem 3.1, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\left\{\omega\operatorname{sinc}\left[\beta(x/y)^{1/2}\right]+(1-\omega)\operatorname{sinc}\left[\rho(x/y)^{1/2}\right]\right\}\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$=\omega\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\operatorname{sinc}\left[\beta(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$+(1-\omega)\int_0^{+\infty}\int_0^{+\infty}\frac{1-\alpha\operatorname{sinc}\left[\rho(x/y)^{1/2}\right]\cos\left[\theta(x/y)^{1/2}\right]}{\gamma x+y}p(x)q(y)dxdy$$
$$\leq \omega D_{\alpha,\beta,\gamma,\theta}+(1-\omega)D_{\alpha,\rho,\gamma,\theta}=G_{\alpha,\beta,\gamma,\rho,\theta,\omega}.$$

The stated inequalities are established. □

This result is just one example of how we can combine Theorems 2.1 and 3.1 to create new integral inequalities. Other possibilities can be considered, including linear convex combinations with three or more components.

4. Conclusion

The HII has been studied extensively in the literature, with several variants and generalizations. However, few of them have considered trigonometric variants while keeping the same form of the upper bound, i.e., "constant multiplied by the product of the L_2 norm of the two main functions (without weight)", especially using the cosine and sine functions. This lack of results has motivated this paper. We have determined two valuable trigonometric generalizations of the HII with four adjustable parameters, and cosine and sine functions. The main constants involved in these inequalities have manageable expressions, making them attractive for use beyond the purposes of the study. Other integral inequalities are also derived. We believe that more can be done in this direction, giving the first step towards possible extensions and variants of the HII to a higher dimension or other trigonometric variant schemes.

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