

**Binary π generalized pre closed sets in binary topological spaces****M. Vinoth¹*, R. Asokan², V. Ramesh³ and R. Premkumar⁴**¹Department of Mathematics, School of Mathematics, Madurai Kamaraj University, Madurai-625 021, Tamil Nadu, India. ORCID iD: [0009-0002-3638-1119](https://orcid.org/0009-0002-3638-1119)²Department of Mathematics, School of Mathematics, Madurai Kamaraj University, Madurai-625 021, Tamil Nadu, India. ORCID iD: [0000-0002-0992-1426](https://orcid.org/0000-0002-0992-1426)³PG and Research Department of Mathematics, Kandaswami Kandar's College, P. Velur, Namakkal-638 182, Tamil Nadu, India. ORCID iD: [0000-0002-1243-6958](https://orcid.org/0000-0002-1243-6958)⁴Department of Mathematics, Arul Anandar College, Karumathur, Madurai-625 514, Tamil Nadu, India. ORCID iD: [0000-0002-8088-5114](https://orcid.org/0000-0002-8088-5114)**Received:** 14 Jun 2024**Accepted:** 13 Aug 2024**Published Online:** 15 Sep 2024

Abstract: In this paper, we introduce and investigate the concepts of binary πgp -closed sets in a binary topological spaces. Also, we characterize the relationships between them and the related properties.

Key words: binary gp -closed set, binary πgp -closed set and binary gpr -set

1. Introduction and Preliminaries

In 2011, Nithyanantha Jothi and Thangavelu [5], they have introduced topology between two sets and also studied some of their properties. Topology between two sets is the binary structure from X to Y which is defined to be the ordered pairs (A, B) where $A \subseteq X$ and $B \subseteq Y$. Recently some author's developed in the modern topologies for example ([1], [3], [4], [9], and [10]). In this paper, we introduce and investigate the ideas of binary πgp -closed sets in a binary topological space. Also we characterize the relations between them and the related properties.

Let X and Y be any two nonempty sets. A binary topology [5] from X to Y is a binary structure $\mathcal{M} \subseteq \mathbb{P}(X) \times \mathbb{P}(Y)$ that satisfies the axioms namely

1. (ϕ, ϕ) and $(X, Y) \in \mathcal{M}$,
2. $(A_1 \cap A_2, B_1 \cap B_2) \in \mathcal{M}$ whenever $(A_1, B_1) \in \mathcal{M}$ and $(A_2, B_2) \in \mathcal{M}$, and
3. If $\{(A_\alpha, B_\alpha) : \alpha \in \delta\}$ is a family of members of \mathcal{M} , then $(\bigcup_{\alpha \in \delta} A_\alpha, \bigcup_{\alpha \in \delta} B_\alpha) \in \mathcal{M}$.

If \mathcal{M} is a binary topology from X to Y then the triplet (X, Y, \mathcal{M}) is called a binary topological space and the members of \mathcal{M} are called the binary open subsets of the binary topological space (X, Y, \mathcal{M}) . The elements of $X \times Y$ are called the binary points of the binary topological space (X, Y, \mathcal{M}) . If $Y = X$ then \mathcal{M} is called a binary topology on X in which case we write (X, \mathcal{M}) as a binary topological space.

Definition 1.1. [5] Let X and Y be any two nonempty sets and let (A, B) and $(C, D) \in \mathbb{P}(X) \times \mathbb{P}(Y)$. We say that $(A, B) \subseteq (C, D)$ if $A \subseteq C$ and $B \subseteq D$.

Definition 1.2. [5] Let (X, Y, \mathcal{M}) be a binary topological space and $A \subseteq X, B \subseteq Y$. Then (A, B) is called binary closed in (X, Y, \mathcal{M}) if $(X \setminus A, Y \setminus B) \in \mathcal{M}$.

Proposition 1.1. [5] Let (X, Y, \mathcal{M}) be a binary topological space and $(A, B) \subseteq (X, Y)$.

Let $(A, B)^{1*} = \cap\{A_\alpha : (A_\alpha, B_\alpha) \text{ is binary closed and } (A, B) \subseteq (A_\alpha, B_\alpha)\}$ and $(A, B)^{2*} = \cap\{B_\alpha : (A_\alpha, B_\alpha) \text{ is binary closed and } (A, B) \subseteq (A_\alpha, B_\alpha)\}$. Then $((A, B)^{1*}, (A, B)^{2*})$ is binary closed and $(A, B) \subseteq ((A, B)^{1*}, (A, B)^{2*})$.

Proposition 1.2. [5] Let (X, Y, \mathcal{M}) be a binary topological space and $(A, B) \subseteq (X, Y)$. Let $(A, B)^{1*} = \cup\{A_\alpha : (A_\alpha, B_\alpha) \text{ is binary open and } (A_\alpha, B_\alpha) \subseteq (A, B)\}$ and $(A, B)^{2*} = \cup\{B_\alpha : (A_\alpha, B_\alpha) \text{ is binary open and } (A_\alpha, B_\alpha) \subseteq (A, B)\}$.

Definition 1.3. [5] The ordered pair $((A, B)^{1*}, (A, B)^{2*})$ is called the binary closure of (A, B) , denoted by $b-cl(A, B)$ in the binary space (X, Y, \mathcal{M}) where $(A, B) \subseteq (X, Y)$.

Definition 1.4. [5] The ordered pair $((A, B)^{1*}, (A, B)^{2*})$ defined in proposition 1.2 is called the binary interior of (A, B) , denoted by $b-int(A, B)$. Here $((A, B)^{1*}, (A, B)^{2*})$ is binary open and $((A, B)^{1*}, (A, B)^{2*}) \subseteq (A, B)$.

Definition 1.5. [5] Let (X, Y, \mathcal{M}) be a binary topological space and let $(x, y) \subseteq (X, Y)$. The binary open set (A, B) is said to be a binary neighbourhood of (x, y) if $x \in A$ and $y \in B$.

Proposition 1.3. [5] Let $(A, B) \subseteq (C, D) \subseteq (X, Y)$ and (X, Y, \mathcal{M}) be a binary topological space. Then, the following statements hold:

1. $b-int(A, B) \subseteq (A, B)$.
2. If (A, B) is binary open, then $b-int(A, B) = (A, B)$.
3. $b-int(A, B) \subseteq b-int(C, D)$.
4. $b-int(b-int(A, B)) = b-int(A, B)$.
5. $(A, B) \subseteq b-cl(A, B)$.
6. If (A, B) is binary closed, then $b-cl(A, B) = (A, B)$.
7. $b-cl(A, B) \subseteq b-cl(C, D)$.
8. $b-cl(b-cl(A, B)) = b-cl(A, B)$.

Definition 1.6. A subset (A, B) of a binary topological space (X, Y, \mathcal{M}) is called

1. a binary semi open set [6] if $(A, B) \subseteq b-cl(b-int(A, B))$.
2. a binary pre open set [2] if $(A, B) \subseteq b-int(b-cl(A, B))$.
3. a binary regular open set [7] if $(A, B) = b-int(b-cl(A, B))$.
4. a binary π -open [11] if the finite union of binary regular-open sets.

The complements of the above mentioned sets are called their respective closed sets.

Definition 1.7. A subset (A, B) of a binary topological space (X, Y, \mathcal{M}) is called

1. a binary g -closed set [8] if $b-cl(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open.
2. a binary πg -closed [11] if $b-cl(A, B) \subseteq (U, V)$, whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary π -open.
3. a binary rg -closed set [7] if $b-cl(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary regular open.

The complements of the above mentioned sets are called their respective open sets.

2. Binary πgp -closed sets

Definition 2.1. A subset (A, B) of (X, Y) in a binary topological space (X, Y, \mathcal{M}) is called a

1. binary generalized pre closed set (briefly, binary gp -closed set) if $b-pcl(A, B) \subseteq (G, H)$ whenever $(A, B) \subseteq (G, H)$ and (G, H) is binary open.
2. binary π generalized pre closed set (briefly, binary πgp -closed set) if $b-pcl(A, B) \subseteq (G, H)$ whenever $(A, B) \subseteq (G, H)$ and (G, H) is binary π -open.

The complements of the above mentioned sets are called their respective open sets.

Example 2.1. Let $X = \{1, 2\}$, $Y = \{a, b\}$ with the binary topology $\mathcal{M} = \{(\phi, \phi), (\phi, \{b\}), (\{1\}, \{a\}), (\{1\}, Y), (X, Y)\}$. Then the binary πgp -closed sets are (ϕ, ϕ) , $(\phi, \{a\})$, $(\{1\}, \phi)$, $(\{2\}, \phi)$, $(\{2\}, \{a\})$, $(\{2\}, \{b\})$, $(\{2\}, Y)$, (X, ϕ) , $(X, \{a\})$, $(X, \{b\})$, (X, Y) . Then the subset $(\{2\}, \{a\})$ is binary πgp -closed set but $(\{1\}, \{a\})$ is not binary πgp -closed set.

Theorem 2.1. In a binary topological space (X, Y, \mathcal{M}) . If (A, B) is binary π -open and binary πgp -closed then binary pre closed and hence binary clopen.

Proof. If (A, B) is binary π -open and binary πgp -closed, then $b-pcl(A, B) \subseteq (A, B)$ and so (A, B) is binary pre closed. Hence (A, B) binary clopen, since binary π -open set is binary open and binary pre open set is binary closed. \square

Proposition 2.1. In a binary topological space (X, Y, \mathcal{M}) . If (A, B) is binary semi open and binary πgp -closed set then (A, B) is binary πg -closed.

Proof. Let $(A, B) \subseteq (G, H)$ and (G, H) be binary π -open. Since (A, B) is binary πgp -closed, $b-pcl(A, B) \subseteq (G, H)$. Since (A, B) is binary semi-open, $b-pcl(A, B) = b-cl(A, B) \subseteq (G, H)$ and hence (A, B) is binary πg -closed. \square

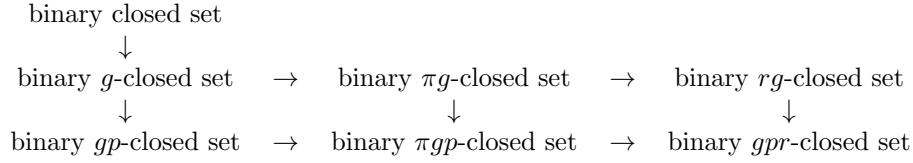
Remark 2.1. The converse of Proposition 2.1 is need not be true as shown in the following Example.

Example 2.2. In Example 2.1, binary πg -closed sets are (ϕ, ϕ) , $(\{2\}, \phi)$, $(\{2\}, \{a\})$, $(\{2\}, \{b\})$, $(\{2\}, Y)$, (X, ϕ) , $(X, \{a\})$, $(X, \{b\})$, (X, Y) , then the subset $(\phi, \{a\})$ is binary πgp -closed set but not binary πg -closed.

Definition 2.2. A subset (A, B) of a binary topological space (X, Y, \mathcal{M}) is called a binary gpr -closed set if $b-pcl(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary regular open.

Example 2.3. In Example 2.1, binary gpr -closed sets are (ϕ, ϕ) , $(\phi, \{a\})$, (ϕ, Y) , $(\{1\}, \phi)$, $(\{1\}, \{b\})$, $(\{1\}, Y)$, $(\{2\}, \phi)$, $(\{2\}, \{a\})$, $(\{2\}, \{b\})$, $(\{2\}, Y)$, (X, ϕ) , $(X, \{a\})$, $(X, \{b\})$, (X, Y) .

Remark 2.2. *The following diagram holds for any subset of (X, Y) in a binary topological space (X, Y, \mathcal{M}) .*



Here $A \rightarrow B$ means A implies B .

Remark 2.3. *None of the implications is reversible as shown in the following Examples.*

Example 2.4. *Let $X = \{a, b\}$, $Y = \{1, 2\}$ with the binary topology $\mathcal{M} = \{(\phi, \phi), (\phi, \{2\}), (\{a\}, \{1\}), (\{a\}, Y), (X, \{1\}), (X, Y)\}$. Then binary πgp -closed sets are $\mathbb{P}(X) \times \mathbb{P}(Y)$ and binary gp -closed sets are $\mathbb{P}(X) \times \mathbb{P}(Y) - ((\{a\}, \{1\}), (\{a\}, Y))$, then the subset $(\{a\}, \{1\})$ is binary πgp -closed set but not binary gp -closed set.*

Example 2.5. *In Example 2.1, binary πg -closed sets are $(\phi, \phi), (\{2\}, \phi), (\{2\}, \{a\}), (\{2\}, \{b\}), (\{2\}, Y), (X, \phi), (X, \{a\}), (X, \{b\}), (X, Y)$, then the subset $(\phi, \{a\})$ is binary πgp -closed set but not binary πg -closed.*

Example 2.6. *In Example 2.1, binary gpr -closed sets are $(\phi, \phi), (\phi, \{a\}), (\phi, Y), (\{1\}, \phi), (\{1\}, \{b\}), (\{1\}, Y), (\{2\}, \phi), (\{2\}, \{a\}), (\{2\}, \{b\}), (\{2\}, Y), (X, \phi), (X, \{a\}), (X, \{b\}), (X, Y)$, the subset $(\{1\}, \{b\})$ is binary gpr -closed set but not binary πgp -closed.*

Remark 2.4. *The following example shows that every binary closed set in (X, Y, \mathcal{M}) , is binary πgp -closed in (X, Y, \mathcal{M}) , but not conversely.*

Example 2.7. *In Example 2.1, then the subset $(\{2\}, Y)$ is binary πgp -closed set but not binary closed.*

Theorem 2.2. *Let (A, B) be a binary πgp -closed set in a binary topological space (X, Y, \mathcal{M}) . Then $b\text{-}pcl(A, B) - (A, B)$ does not contain any non-empty binary π -closed set.*

Proof. Let (C, D) be a binary π -closed set such that $(C, D) \subseteq b\text{-}pcl(A, B) - (A, B)$. Then $(A, B) \subseteq (G, H) - (C, D)$. Since (A, B) is binary πgp -closed and $(G, H) - (C, D)$ is binary π -open, $b\text{-}pcl(A, B) \subseteq (G, H) - (C, D)$, i.e., $(C, D) \subseteq (G, H) - b\text{-}pcl(A, B)$. Hence $(C, D) \subseteq b\text{-}pcl(A, B) \cap ((G, H) - b\text{-}pcl(A, B)) = (\phi, \phi)$. This follows that $(C, D) = (\phi, \phi)$. \square

Corollary 2.1. *Let (X, Y, \mathcal{M}) be a binary topological space. Then a subset of (X, Y) is binary pre closed if and only if it is both $b\text{-}pcl(A, B) - (A, B)$ is binary π -closed and $(A, B) = b\text{-}pcl(b\text{-}int(A, B))$.*

Proof. Obvious. \square

Theorem 2.3. *The union of two binary πgp -closed sets is binary πgp -closed in a binary topological space (X, Y, \mathcal{M}) .*

Proof. Let $(A, B) \cup (C, D) \subseteq (G, H)$, then $(A, B) \subseteq (G, H)$ and $(C, D) \subseteq (G, H)$ where (G, H) is binary π -open. As (A, B) and (C, D) are πgp -closed, $b\text{-}cl(A, B) \subseteq (G, H)$ and $b\text{-}cl(C, D) \subseteq (G, H)$. Hence $b\text{-}cl((A, B) \cup (C, D)) = b\text{-}cl(A, B) \cup b\text{-}cl(C, D) \subseteq (G, H)$. \square

Remark 2.5. *The following example shows that intersection of two binary πgp -closed sets is binary πgp -closed in a binary topological space (X, Y, \mathcal{M}) .*

Example 2.8. *In Example 2.1, then $(A, B) = (\{2\}, \{a\})$ and $(C, D) = (X, \phi)$ are binary πgp -closed. Clearly $(A, B) \cap (C, D) = (\{2\}, \phi)$ is binary πgp -closed.*

Corollary 2.2. *In a binary topological space (X, Y, \mathcal{M}) , if (A, B) is binary πgp -closed and binary regular open and (C, D) is binary pre closed in (X, Y) , then $(A, B) \cap (C, D)$ is binary πgp -closed.*

Proof. Let $(A, B) \cap (C, D) \subseteq (G, H)$ and (G, H) is binary π -open in (A, B) . Since (C, D) is binary pre-closed in (X, Y) , $(A, B) \cap (C, D)$ is binary pre-closed in (A, B) and so $b-pcl_{(A,B)}(A, B) \cap (C, D) = (A, B) \cap (C, D)$. That is $b-pcl_{(A,B)}(A, B) \cap (C, D) \subseteq (G, H)$. Then $(A, B) \cap (C, D)$ is binary πgp -closed in the binary πgp -closed and binary regular open set (A, B) and hence $(A, B) \cap (C, D)$ is binary πgp -closed in (X, Y, \mathcal{M}) . \square

Theorem 2.4. *If (A, B) is binary πgp -closed in (X, Y) and $(A, B) \subseteq (C, D) \subseteq b-pcl(A, B)$, then (C, D) is binary πgp -closed.*

Proof. Let $(C, D) \subseteq (G, H)$ and (G, H) be a binary π -open set in (X, Y) . Since $(A, B) \subseteq (G, H)$ and (C, D) is binary πgp -closed, $b-pcl(A, B) \subseteq (G, H)$ and then $b-pcl(C, D) = b-pcl(A, B) \subseteq (G, H)$. Hence (C, D) is binary πgp -closed. \square

Theorem 2.5. *A set (A, B) of a binary topological space (X, Y, \mathcal{M}) is said to be a binary πgp -open sets if and only if $(C, D) \subseteq b-pint(A, B)$ whenever (C, D) is binary π -closed and $(C, D) \subseteq (A, B)$.*

Proof. Obvious. \square

Theorem 2.6. *A subset (A, B) of a binary topological space (X, Y) is binary πgp -open if and only if it is both $(G, H) = (X, Y)$ whenever (G, H) is binary π -open and $b-pint(A, B) \cup ((X, Y) - (A, B)) \subseteq (G, H)$.*

Proof. Let (G, H) be a binary π -open set and $b-pint(A, B) \cup ((X, Y) - (A, B)) \subseteq (G, H)$. Then $(X, Y) - (G, H) \subseteq ((X, Y) - b-pint(A, B)) \cap (A, B)$, i.e., $((X, Y) - (G, H)) \subseteq b-pcl((X, Y) - (A, B)) - ((X, Y) - (A, B))$. Since $(X, Y) - (A, B)$ is binary πgp -closed, by Theorem 2.2, $((X, Y) - (G, H)) = (\phi, \phi)$ and hence $(G, H) = (X, Y)$.

Conversely, let (C, D) be a binary π -open set of (X, Y) and $(C, D) \subseteq (A, B)$. Since $b-pint(A, B) \cup ((X, Y) - (A, B)) = b-int(b-cl(A, B)) \subseteq ((X, Y) - (C, D))$ is binary π -open and $b-pint(A, B) \cup ((X, Y) - (A, B)) \subseteq b-pint(A, B) \cup ((X, Y) - (C, D))$, by hypothesis, $b-pint(A, B) \cup ((X, Y) - (C, D)) = (X, Y)$ and hence $(C, D) \subseteq b-pint(A, B)$. \square

Theorem 2.7. *Let $(A, B) \subseteq (C, D) \subseteq (U, V)$ and (C, D) binary π -open and binary closed in (X, Y) . If (A, B) is binary πgp -open in (C, D) , then (A, B) is binary πgp -open in (X, Y) .*

Proof. Let (M, N) be any binary π -closed set and $(M, N) \subseteq (A, B)$. Since (M, N) is binary π -closed in (C, D) and (A, B) is binary πgp -open in (C, D) , $(M, N) \subseteq b-pint_{(C,D)}(A, B)$ and then $(C, D) \subseteq b-pint_{(U,V)}(A, B) \cap (C, D)$. Hence $(C, D) \subseteq b-pint_{(U,V)}(A, B)$ and so (A, B) is binary πgp -open in (X, Y) . \square

Theorem 2.8. *If (A, B) is binary πgp -open in (X, Y) and $b-pint(A, B) \subseteq (P, Q) \subseteq (A, B)$, then (P, Q) is binary πgp -open.*

Proof. Let $(C, D) \subseteq (P, Q)$ and (C, D) be binary π -closed in (X, Y) . Since (A, B) is binary πgp -open and $(C, D) \subseteq (A, B)$. $(C, D) \subseteq b-pint(A, B)$ and then $(C, D) \subseteq b-pint(P, Q)$. Thus proves that (P, Q) is binary πgp -open. \square

Theorem 2.9. *A subset (A, B) of (X, Y) is binary πgp -closed if and only if $b-pcl(A, B) - (A, B)$ is binary πgp -open.*

Proof. Let $(C, D) \subseteq b-pcl(A, B) - (A, B)$ and (C, D) be binary π -closed in (X, Y) . Then by Theorem 2.2, $(C, D) = (\phi, \phi)$ and so $(C, D) \subseteq b-pint(b-pcl(A, B) - (A, B))$. This verifies that $b-pcl(A, B) - (A, B)$ is binary πgp -open.

Conversely, let (G, H) be a binary π -open set of (X, Y) and $(A, B) \subseteq (G, H)$. Then $b-pcl(A, B) \cap ((X, Y) - (G, H)) = b-cl(b-int(A, B)) \cap ((X, Y) - (G, H))$ is binary π -closed set contained in $b-pcl(A, B) - (A, B)$. Since $b-pcl(A, B) - (A, B)$ is binary πgp -open, by Theorem 2.5, $b-pcl(A, B) \cap ((X, Y) - (G, H)) \subseteq b-pint(b-pcl(A, B) - (A, B)) = (\phi, \phi)$ and hence $b-pcl(A, B) \subseteq (G, H)$. \square

3. Conclusion

The main aim of this paper is to introduce and investigate the ideas of binary πgp -closed sets in binary topological spaces. Also, we characterize the relations between them and the related properties with suitable examples. In the future, we can study in the area of πgp -irresolute functions (i.e., the domain and the co-domain are the binary topological space) and as well as πgp -open map, πgp -closed map. We can extend by finding an inverse image or image for the πgp -closed set to πgp -closed set also. And we can learn in various areas of topological space with associated applications.

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