Asia Mathematika
Volume: 8 Issue: 2, (2024) Pages: 1 – 15
Available online at **www.asiamath.org**

Integral representations of the Jacobsthal and Jacobsthal-Lucas numbers

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Received: 27 May 2024 • **Accepted:** 09 Aug 2024 • **Published Online:** 15 Sep 2024

Abstract: Our study aims to obtain integral representations of Jacobsthal $\mathcal{J}_{kn}$ and Jacobsthal-Lucas $\mathcal{L}_{kn}$, and then to use these integral representations to derive integral representations of Jacobsthal $\mathcal{J}_{kn+r}$ and Jacobsthal-Lucas $\mathcal{L}_{kn+r}$, where $n \in \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \ldots\}$ is a non-negative integer, $k \in \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer.

Key words: Integral representation; Jacobsthal number; Jacobsthal-Lucas number

1. Introduction

Numerous studies use integral representations of special numbers obtained from different counting sequences as a tool. The importance of obtaining integral representations of different special numbers is illustrated by this fact.

There has been considerable interest in these integral representations of special numbers. Several papers have been devoted to the study of the integral representations of some special numbers. It has been demonstrated in the literature that integral representations of special numbers can be obtained using standard or advanced mathematics from the integral calculus.

Several recent developments have been made in integral representations of special numbers arising from different counting sequences.

Many studies have been conducted on the representation of Catalan numbers as integrals.

[10] and [13], we recall that the Catalan numbers C_n are defined by

$$C_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}, \quad n = 0, 1, 2, \ldots.$$

Using the properties of a combinatorial system, Dana-Picard [4] showed that a Catalan number can be defined in a variety of ways. Additionally, Dana-Picard presented integral representations of these Catalan numbers in that paper.

Using two multiparameter families of definite integrals, Dana-Picard and Zeitoun [5] computed closed forms. As a result, they obtained combinatorial formulas.

In [6], Dana-Picard derived a combinatorial identity for the Catalan numbers as well as two integral representations of them.

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By searching for closed forms of integrals depending on a parameter, Dana-Picard [7] obtained integral identities.

In [8], Dana-Picard and Zeitoun found that a closed formula could be computed for improper integrals using the Wallis formula and a non-straightforward recurrence formula. Hence, a new integral representation is provided for Catalan numbers.

By using the Mellin transform, Penson and Sixdeniers [14] developed an integral representation for Catalan numbers.

Recall from [1] that the Fibonacci numbers $F_n, \quad n = 0, 1, 2, \ldots,$ are defined by $F_0 = 0, F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n, \quad n = 0, 1, 2, \ldots$$

and Lucas numbers $L_n, \quad n = 0, 1, 2, \ldots,$ are defined by $L_0 = 2, L_1 = 1$ and

$$L_{n+2} = L_{n+1} + L_n, \quad n = 0, 1, 2, \ldots.$$

In their publication [9], Glasser and Zhou presented an integral representation of the Fibonacci numbers. In [15], Stewart gave integral representations of Fibonacci and Lucas numbers.

Recall from [12] that the Motzkin numbers $M_n, \quad n = 0, 1, 2, \ldots,$ are defined by

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k, \quad n = 0, 1, 2, \ldots.$$

The integral representation of Motzkin numbers was developed by Mccalla and Nkwanta [11].

Recall from [2], [3] and [16] that the Jacobsthal numbers $\mathcal{J}_n, \quad n = 0, 1, 2, \ldots,$ are defined by $\mathcal{J}_0 = 0, \mathcal{J}_1 = 1$ and

$$\mathcal{J}_{n+2} = \mathcal{J}_{n+1} + 2\mathcal{J}_n, \quad n = 0, 1, 2, \ldots$$

and the formula of the general term is given by

$$\mathcal{J}_n = \frac{1}{3}\left[2^n - (-1)^n\right]. \tag{1}$$

Recall from [2], [3] and [16] that the Jacobsthal-Lucas numbers $\mathcal{L}_n, \quad n = 0, 1, 2, \ldots,$ are defined by $\mathcal{L}_0 = 2, \mathcal{L}_1 = 1$ and

$$\mathcal{L}_{n+2} = \mathcal{L}_{n+1} + 2\mathcal{L}_n, \quad n = 0, 1, 2, \ldots$$

and the formula of the general term is given by

$$\mathcal{L}_n = 2^n + (-1)^n. \tag{2}$$

There are a number of interesting mathematical facts and theorems associated with the Jacobsthal numbers $\mathcal{J}_n$ and Jacobsthal-Lucas numbers $\mathcal{L}_n$ that can be applied to a wide range of problems today. Please refer to [2], [3] and [16] and closely related references for information on new developments of these types of numbers.

Jacobsthal numbers have practical applications in areas such as computer science, cryptography, and combinatorial design. For instance, they can be used in algorithms for efficient data encoding and decoding.

Additionally, Jacobsthal numbers are useful in the construction of certain types of error-correcting codes, which are essential for reliable data transmission.

The integral representation of Jacobsthal numbers allows us to calculate the values of Jacobsthal numbers explicitly and to approximate them asymptotically. This representation is also useful in understanding the properties of Jacobsthal numbers. This representation can also be used to find the upper and lower bounds for the Jacobsthal numbers. Additionally, this representation can be used to study the growth of Jacobsthal numbers.

In this study, we will obtain integral representations of Jacobsthal $\mathcal{J}_{kn}$ and Jacobsthal-Lucas $\mathcal{L}_{kn}$, and from these integral representations we will derive integral representations of Jacobsthal $\mathcal{J}_{kn+r}$ and Jacobsthal-Lucas $\mathcal{L}_{kn+r}$, where $n \in \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \ldots\}$ is a non-negative integer, $k \in \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer.

The following section presents several facts concerning Jacobsthal and Jacobsthal-Lucas numbers.

2. Preliminaries

This section examines some key facts about Jacobsthal and Jacobsthal-Lucas numbers.

Let $\alpha = 2$. From (1), it follows that

$$\mathcal{J}_n = \frac{1}{3}\left(\alpha^n - (-1)^n\right), \tag{3}$$

called Binet's formula for the Jacobsthal numbers and from (2), it follows that

$$\mathcal{L}_n = \alpha^n + (-1)^n, \tag{4}$$

called Binet's formula for the Jacobsthal-Lucas numbers.

α and these two types of numbers have the following relationships.

1. Using (3) and (4), we find that the connection between the Jacobsthal numbers, the Jacobsthal-Lucas numbers, and α is for $n \in \mathbb{Z}_{\geqslant 0}$

$$\alpha^n = \frac{\mathcal{L}_n + 3\mathcal{J}_n}{2}. \tag{5}$$

2. In order to establish the connection between the Jacobsthal numbers and the Jacobsthal-Lucas numbers, straightforward computation yields from (3) and (4) for $n \in \mathbb{Z}_{\geqslant 0}$

$$\begin{aligned}
\mathcal{L}_n^2 - 9\mathcal{J}_n^2 &= \left(\alpha^n + (-1)^n\right)^2 - 9\left(\frac{1}{3}\left(\alpha^n - (-1)^n\right)\right)^2 \\
&= \left(\alpha^{2n} + (-1)^{2n} + 2\alpha^n(-1)^n\right) \\
&\quad - 9\left(\frac{1}{9}\left(\alpha^{2n} + (-1)^{2n} - 2\alpha^n(-1)^n\right)\right) \\
&= 4\alpha^n(-1)^n \\
&= 2^{n+2}(-1)^n.
\end{aligned} \tag{6}$$

3. By direct calculation, we determine the Jacobsthal index addition formulae for $m, r \in \mathbb{Z}_{\geqslant 0}$, from (3) and (4)

$$
\begin{aligned}
\mathcal{J}_r \mathcal{L}_m + \mathcal{L}_r \mathcal{J}_m &= \frac{1}{3}\left(\alpha^r - (-1)^r\right)\left(\alpha^m + (-1)^m\right) \\
&\quad + \frac{1}{3}\left(\alpha^r + (-1)^r\right)\left(\alpha^m - (-1)^m\right) \\
&= \frac{1}{3}\left(\alpha^{r+m} + \alpha^r(-1)^m - \alpha^m(-1)^r - (-1)^{r+m}\right) \\
&\quad + \frac{1}{3}\left(\alpha^{r+m} - \alpha^r(-1)^m + \alpha^m(-1)^r - (-1)^{r+m}\right) \\
&= \frac{2}{3}\left(\alpha^{r+m} - (-1)^{r+m}\right) \\
&= 2\mathcal{J}_{m+r}.
\end{aligned} \tag{7}
$$

4. The Jacobsthal-Lucas index addition formulae can be derived directly from (3) and (4) for $m, r \in \mathbb{Z}_{\geqslant 0}$

$$
\begin{aligned}
\mathcal{L}_m \mathcal{L}_r + 9\mathcal{J}_m \mathcal{J}_r &= \left(\alpha^m + (-1)^m\right)\left(\alpha^r + (-1)^r\right) \\
&\quad + 9\left(\frac{1}{3}\right)^2\left(\alpha^m - (-1)^m\right)\left(\alpha^r - (-1)^r\right) \\
&= \alpha^{m+r} + (-1)^r\alpha^m + (-1)^m\alpha^r + (-1)^{m+r} \\
&\quad + \alpha^{m+r} - (-1)^r\alpha^m - (-1)^m\alpha^r + (-1)^{m+r} \\
&= 2\left(\alpha^{m+r} + (-1)^{m+r}\right) \\
&= 2\mathcal{L}_{m+r}.
\end{aligned} \tag{8}
$$

3. Integral representations for the Jacobsthal Numbers $\mathcal{J}_{kn}$ and the Jacobsthal-Lucas Numbers $\mathcal{L}_{kn}$

The purpose of this section is to present integral representations of Jacobsthal numbers $\mathcal{J}_{kn}$ and for the Jacobsthal-Lucas numbers $\mathcal{L}_{kn}$, where $n \in \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, \ldots\}$ is a non-negative integer and $k \in \mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ is an arbitrary but fixed positive integer.

The following theorem gives an integral representation of the Jacobsthal numbers $\mathcal{J}_{kn}$. Based on this theorem, we derive an integral representation of the Jacobsthal numbers $\mathcal{J}_n$, an integral representation of the Jacobsthal numbers $\mathcal{J}_{2n}$ with even integer indexes, an integral representation of the Jacobsthal numbers $\mathcal{J}_{2n+1}$ with odd integer index and Binet's formula for $\mathcal{J}_{kn}$.

Theorem 3.1. *We have an integral representation of the Jacobsthal numbers $\mathcal{J}_{kn}$ by the integral*

$$
\mathcal{J}_{kn} = \frac{n\mathcal{J}_k}{2^n} \int_{-1}^{1} \left(\mathcal{L}_k + 3\mathcal{J}_k x\right)^{n-1} dx \tag{9}
$$

for $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$.

Proof. Let I be the integral to be found. We make the substitution

$$u = g(x) = \mathcal{L}_k + 3\mathcal{J}_k x$$

because its differential is $du = 3\mathcal{J}_k dx$, which, apart from the factor $3\mathcal{J}_k$, occurs in the integral. Then, we obtain $dx = \frac{1}{3\mathcal{J}_k} du$. Before substituting, determine the new upper and lower limits of integration. When $x = -1$, the new lower limit is $u = g(-1)$ and when $x = 1$, the new upper limit is $u = g(1)$. Now, we can substitute to obtain

$$\begin{aligned}
\frac{n\mathcal{J}_k}{2^n} I &= \frac{n\mathcal{J}_k}{2^n} \int_{-1}^{1} (\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1} dx \\
&= \frac{n\mathcal{J}_k}{2^n} \frac{1}{3\mathcal{J}_k} \int_{g(-1)}^{g(1)} u^{n-1} du \\
&= \frac{1}{3} \frac{n}{2^n} \frac{1}{n} \left[u^n\right]_{g(-1)}^{g(1)} \\
&= \frac{1}{3} \frac{1}{2^n} \left[(\mathcal{L}_k + 3\mathcal{J}_k x)^n\right]_{-1}^{1} \\
&= \frac{1}{3} \left[\left(\frac{\mathcal{L}_k + 3\mathcal{J}_k x}{2}\right)^n\right]_{-1}^{1} \\
&= \frac{1}{3} \left[\left(\frac{\mathcal{L}_k + 3\mathcal{J}_k}{2}\right)^n - \left(\frac{\mathcal{L}_k - 3\mathcal{J}_k}{2}\right)^n\right]. \qquad (10)
\end{aligned}$$

From (5) and (6), direct calculation gives

$$\begin{aligned}
\frac{1}{\alpha^n} &= \frac{2}{\mathcal{L}_n + 3\mathcal{J}_n} \\
&= \frac{2(\mathcal{L}_n - 3\mathcal{J}_n)}{(\mathcal{L}_n + 3\mathcal{J}_n)(\mathcal{L}_n - 3\mathcal{J}_n)} \\
&= \frac{2(\mathcal{L}_n - 3\mathcal{J}_n)}{\mathcal{L}_n^2 - 9\mathcal{J}_n^2} \\
&= \frac{2}{2^{n+2}(-1)^n} (\mathcal{L}_n - 3\mathcal{J}_n) \\
&= \frac{(-1)^n}{2^{n+1}} (\mathcal{L}_n - 3\mathcal{J}_n).
\end{aligned}$$

Hence, we have

$$(-1)^n = \frac{\mathcal{L}_n - 3\mathcal{J}_n}{2}. \qquad (11)$$

From (10) and (11), it follows that

$$\begin{aligned}\frac{n\mathcal{J}_k}{2^n} I &= \frac{1}{3}\left[\left(\alpha^k\right)^n - \left((-1)^k\right)^n\right] \\ &= \frac{1}{3}\left[\alpha^{kn} - (-1)^{kn}\right] \\ &= \mathcal{J}_{kn}.\end{aligned}$$

Thus, the proof of Theorem 3.1 is completed. □

In the following corollary, an integral representation of the Jacobsthal numbers $\mathcal{J}_n$ is represented.

Corollary 3.1. *We have an integral representation of the Jacobsthal numbers $\mathcal{J}_n$ by the integral*

$$\mathcal{J}_n = \frac{n}{2^n} \int_{-1}^{1} (1+3x)^{n-1}\, dx$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. If we write $k = 1$ in (9), then we obtain the integral representations of Jacobsthal numbers $\mathcal{J}_n$ as follows:

$$\begin{aligned}\mathcal{J}_n &= \frac{n\mathcal{J}_1}{2^n} \int_{-1}^{1} (\mathcal{L}_1 + 3x\mathcal{J}_1)^{n-1}\, dx \\ &= \frac{n}{2^n} \int_{-1}^{1} (1+3x)^{n-1}\, dx.\end{aligned}$$

Thus, the proof of Corollary 3.1 is completed. □

In the following corollary, an integral representation of the Jacobsthal numbers with even integer indexes is represented.

Corollary 3.2. *We have an integral representation of the Jacobsthal numbers $\mathcal{J}_{2n}$ by the integral*

$$\mathcal{J}_{2n} = \frac{n}{2^n} \int_{-1}^{1} (5+3x)^{n-1}\, dx. \tag{12}$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. If we set $k = 2$ in (9), then we get an integral representation of the Jacobsthal numbers with even integer index by

$$\begin{aligned}\mathcal{J}_{2n} &= \frac{n\mathcal{J}_2}{2^n} \int_{-1}^{1} (\mathcal{L}_2 + 3x\mathcal{J}_2)^{n-1}\, dx \\ &= \frac{n}{2^n} \int_{-1}^{1} (5+3x)^{n-1}\, dx.\end{aligned}$$

The proof of Corollary 3.2 is complete. □

The following corollary gives an integral representation of the Jacobsthal numbers with odd integer index.

Corollary 3.3. *We have an integral representation of the Jacobsthal numbers* $\mathcal{J}_{2n+1}$ *by the integral*

$$\mathcal{J}_{2n+1} = \frac{1}{2^{n+1}} \int_{-1}^{1} (5 + n + 3(n+1)x)(5+3x)^{n-1}\,dx \tag{13}$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. We first recall the obvious identity $\mathcal{J}_{2n+2} = \mathcal{J}_{2n+1} + 2\mathcal{J}_{2n}$. Then, from this identity, it follows that

$$\mathcal{J}_{2n+1} = \mathcal{J}_{2n+2} - 2\mathcal{J}_{2n}. \tag{14}$$

Using a reindexing of $n \mapsto n+1$ in (12), from (12) and (14) straightforward computation yields

$$\begin{aligned}
\mathcal{J}_{2n+1} &= \mathcal{J}_{2n+2} - 2\mathcal{J}_{2n} \\
&= \frac{n+1}{2^{n+1}} \int_{-1}^{1} (5+3x)^{n}\,dx - 2\frac{n}{2^{n}} \int_{-1}^{1} (5+3x)^{n-1}\,dx \\
&= \frac{1}{2^{n+1}} \int_{-1}^{1} [(n+1)(5+3x) - 4n](5+3x)^{n-1}\,dx \\
&= \frac{1}{2^{n+1}} \int_{-1}^{1} [5n + 5 + 3(n+1)x - 4n](5+3x)^{n-1}\,dx \\
&= \frac{1}{2^{n+1}} \int_{-1}^{1} (5 + n + 3(n+1)x)(5+3x)^{n-1}\,dx.
\end{aligned}$$

The proof of Corollary 3.3 is complete. □

The following corollary gives a thinly disguised form of Binet's formula for $\mathcal{J}_{kn}$.

Corollary 3.4. *The Jacobsthal numbers* $\mathcal{J}_{kn}$ *can be represented by*

$$\mathcal{J}_{kn} = \frac{n}{3} \int_{(-1)^k}^{\alpha^k} t^{n-1}dt$$

for $n \in \mathbb{Z}_{\geqslant 0}$ *and arbitrary but fixed* $k \in \mathbb{Z}_{>0}$.

Proof. Using the substitution $t = \frac{1}{2}(\mathcal{L}_k + 3\mathcal{J}_k x)$ in (9), we have $dt = \frac{3\mathcal{J}_k}{2}dx$ and $dx = \frac{2}{3\mathcal{J}_k}dt$. To find the new limits of integration (9) we note that when $x = -1$,

$$t = \frac{1}{2}(\mathcal{L}_k - 3\mathcal{J}_k) = (-1)^k$$

and when $x=1$,

$$t=\frac{1}{2}\left(\mathcal{L}_k+3\mathcal{J}_k\right)=\alpha^k.$$

Therefore, from (9) we obtain

$$\begin{aligned}\mathcal{J}_{kn}&=\frac{n\mathcal{J}_k}{2^n}\int\limits_{-1}^{1}\left(\mathcal{L}_k+3\mathcal{J}_kx\right)^{n-1}dx\\&=\frac{n\mathcal{J}_k}{2^n}\int\limits_{(-1)^k}^{\alpha^k}\left(2t\right)^{n-1}\frac{2}{3\mathcal{J}_k}dt\\&=\frac{n\mathcal{J}_k}{2^n}2^{n-1}\frac{2}{3\mathcal{J}_k}\int\limits_{(-1)^k}^{\alpha^k}\left(t\right)^{n-1}dt\\&=\frac{n}{3}\int\limits_{(-1)^k}^{\alpha^k}t^{n-1}dt.\end{aligned}$$

The proof of Corollary 3.4 is complete. □

The following theorem gives an integral representation of the Jacobsthal-Lucas numbers.

Theorem 3.2. *We have an integral representation of the Jacobsthal-Lucas numbers $\mathcal{L}_{kn}$ by the integral*

$$\mathcal{L}_{kn}=\frac{1}{2^n}\int\limits_{-1}^{1}\left(\mathcal{L}_k+3\mathcal{J}_k(n+1)x\right)\left(\mathcal{L}_k+3\mathcal{J}_kx\right)^{n-1}dx \tag{15}$$

for $n\in\mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k\in\mathbb{Z}_{>0}$.

Proof. Let

$$I_1=\int\left(\mathcal{L}_k+3\mathcal{J}_k(n+1)x\right)\left(\mathcal{L}_k+3\mathcal{J}_kx\right)^{n-1}dx.$$

To evaluate this integral we use the integration by parts. Let

$$u=\mathcal{L}_k+3\mathcal{J}_k(n+1)x$$

and

$$dv=\left(\mathcal{L}_k+3\mathcal{J}_kx\right)^{n-1}dx.$$

Then,

$$du=3\mathcal{J}_k(n+1)dx$$

and

$$v=\int\left(\mathcal{L}_k+3\mathcal{J}_kx\right)^{n-1}dx.$$

To evaluate the integral that we have obtained, $v = \int (\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1}\, dx$, if we let $t = \mathcal{L}_k + 3\mathcal{J}_k x$, then $dt = 3\mathcal{J}_k dx$, so $dx = \frac{1}{3\mathcal{J}_k} dt$. Therefore,

$$\begin{aligned} v &= \int \frac{1}{3\mathcal{J}_k} t^{n-1} dt \\ &= \frac{1}{3n\mathcal{J}_k} t^n \\ &= \frac{1}{3n\mathcal{J}_k} (\mathcal{L}_k + 3\mathcal{J}_k x)^n . \end{aligned}$$

If we let

$$I_2 = \frac{1}{2^n} \int_{-1}^{1} (\mathcal{L}_k + 3\mathcal{J}_k (n+1) x) (\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1}\, dx,$$

then we obtain

$$\begin{aligned} I_2 &= \frac{1}{2^n} \left\{ [uv]_{-1}^{1} - \int_{-1}^{1} v du \right\} \\ &= \frac{1}{2^n} \left\{ \frac{1}{3n\mathcal{J}_k} \left[(\mathcal{L}_k + 3\mathcal{J}_k (n+1) x) (\mathcal{L}_k + 3\mathcal{J}_k x)^n \right]_{-1}^{1} \right. \\ &\quad \left. -3\mathcal{J}_k (n+1) \frac{1}{3n\mathcal{J}_k} \int_{-1}^{1} (\mathcal{L}_k + 3\mathcal{J}_k x)^n\, dx \right\} \\ &= \frac{1}{3n\mathcal{J}_k} \left(\frac{\mathcal{L}_k + 3\mathcal{J}_k}{2} \right)^n (\mathcal{L}_k + 3\mathcal{J}_k (n+1)) \qquad (16) \\ &\quad - \frac{1}{3n\mathcal{J}_k} \left(\frac{\mathcal{L}_k - 3\mathcal{J}_k}{2} \right)^n (\mathcal{L}_k - 3\mathcal{J}_k (n+1)) \\ &\quad - \frac{n+1}{n2^n} \int_{-1}^{1} (\mathcal{L}_k + 3\mathcal{J}_k x)^n\, dx. \end{aligned}$$

From (9), we have that

$$\mathcal{J}_{k(n+1)} \frac{2^{n+1}}{(n+1)\,\mathcal{J}_k} = \int_{-1}^{1} (\mathcal{L}_k + 3\mathcal{J}_k x)^n\, dx. \qquad (17)$$

Hence, from (16) and (17) we get

$$\begin{aligned} I_2 &= \frac{1}{3n\mathcal{J}_k} \left(\frac{\mathcal{L}_k + 3\mathcal{J}_k}{2} \right)^n (\mathcal{L}_k + 3\mathcal{J}_k (n+1)) \\ &\quad - \frac{1}{3n\mathcal{J}_k} \left(\frac{\mathcal{L}_k - 3\mathcal{J}_k}{2} \right)^n (\mathcal{L}_k - 3\mathcal{J}_k (n+1)) \\ &\quad - \frac{n+1}{n2^n} \frac{2^{n+1}}{(n+1)\,\mathcal{J}_k} \mathcal{J}_{kn+k}. \end{aligned}$$

By (5), (7) and (11), it follows that

$$\begin{aligned}
I_2 &= \frac{1}{3n\mathcal{J}_k}\alpha^{kn}\left(\mathcal{L}_k+3\mathcal{J}_k(n+1)\right) \\
&\quad -\frac{1}{3n\mathcal{J}_k}(-1)^{kn}\left(\mathcal{L}_k-3\mathcal{J}_k(n+1)\right) \\
&\quad -\frac{2}{n\mathcal{J}_k}\mathcal{J}_{kn+k} \\
&= \frac{1}{n\mathcal{J}_k}\left[\frac{1}{3}\left(\alpha^{kn}-(-1)^{kn}\right)\mathcal{L}_k\right. \\
&\quad +\left(\alpha^{kn}+(-1)^{kn}\right)(n+1)\mathcal{J}_k \\
&\quad \left.-2\mathcal{J}_{kn+k}\right] \\
&= \frac{1}{n\mathcal{J}_k}\left[\mathcal{L}_k\mathcal{J}_{kn}+(n+1)\mathcal{J}_k\mathcal{L}_{kn}-2\mathcal{J}_{kn+k}\right] \\
&= \frac{1}{n\mathcal{J}_k}\left[n\mathcal{J}_k\mathcal{L}_{kn}+\mathcal{L}_k\mathcal{J}_{kn}+\mathcal{J}_k\mathcal{L}_{kn}-2\mathcal{J}_{kn+k}\right]. \qquad (18)
\end{aligned}$$

We can verify the following equation by recalling the formula given in (7) and substituting k for r and kn for m:

$$\mathcal{L}_k\mathcal{J}_{kn}+\mathcal{J}_k\mathcal{L}_{kn}-2\mathcal{J}_{kn+k}=0.$$

As a result, (18) provides us the result $I_2=\mathcal{L}_{kn}$. As a consequence, the theorem can be proved by obtaining the result in (15), which concludes the proof of the theorem. □

Corollary 3.5. *We have an integral representation of the Jacobsthal-Lucas numbers* $\mathcal{L}_n$ *by the integral*

$$\mathcal{L}_n=\frac{1}{2^n}\int_{-1}^{1}\left(1+3(n+1)x\right)\left(1+3x\right)^{n-1}dx$$

for $n\in\mathbb{Z}_{\geqslant 0}$.

Proof. As a result of writing $k=1$ at (15), we obtain integral representations for Jacobsthal-Lucas numbers $\mathcal{L}_n$ in the following way:

$$\begin{aligned}
\mathcal{L}_n &= \frac{1}{2^n}\int_{-1}^{1}\left(\mathcal{L}_1+3\mathcal{J}_1(n+1)x\right)\left(\mathcal{L}_1+3\mathcal{J}_1x\right)^{n-1}dx \\
&= \frac{1}{2^n}\int_{-1}^{1}\left(1+3(n+1)x\right)\left(1+3x\right)^{n-1}dx
\end{aligned}$$

The proof of Corollary 3.5 is completed. □

The following corollary gives an integral representation of the Jacobsthal-Lucas numbers with even integer index.

Corollary 3.6. *We have an integral representation of the Jacobsthal-Lucas numbers* $\mathcal{L}_{2n}$ *by the integral*

$$\mathcal{L}_{2n} = \frac{1}{2^n} \int_{-1}^{1} (5 + 3(n+1)x)(5+3x)^{n-1}\, dx \tag{19}$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. If we set $k = 2$ in (15), then we get an integral representation of the even Jacobsthal-Lucas numbers by

$$\begin{aligned} \mathcal{L}_{2n} &= \frac{1}{2^n} \int_{-1}^{1} (\mathcal{L}_2 + 3\mathcal{J}_2(n+1)x)(\mathcal{L}_2 + 3\mathcal{J}_2 x)^{n-1}\, dx \\ &= \frac{1}{2^n} \int_{-1}^{1} (5 + 3(n+1)x)(5+3x)^{n-1}\, dx. \end{aligned}$$

The proof of Corollary 3.6 is complete. □

The following corollary gives an integral representation of the Pell-Luas numbers with even integer index.

Corollary 3.7. *We have an integral representation of the Jacobsthal-Lucas numbers* $\mathcal{L}_{2n+1}$ *by the integral*

$$\mathcal{L}_{2n+1} = \frac{1}{2^{n+1}} \int_{-1}^{1} (5 + 9n + 3(n+1)x)(5+3x)^{n-1}\, dx \tag{20}$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. Recalling that the identity in (80) takes the form of

$$2\mathcal{L}_{m+r} = \mathcal{L}_m \mathcal{L}_r + 9\mathcal{J}_m \mathcal{J}_r,$$

by substituting $2n$ for m and 1 for r, we get the following identity

$$\begin{aligned} 2\mathcal{L}_{2n+1} &= \mathcal{L}_{2n}\mathcal{L}_1 + 9\mathcal{J}_{2n}\mathcal{J}_1 \\ &= \frac{1}{2}(\mathcal{L}_{2n} + 9\mathcal{J}_{2n}). \end{aligned} \tag{21}$$

Substituting the integral representations obtained for $\mathcal{L}_{2n}$ and $\mathcal{J}_{2n}$ in 21 results in the following integral representation for $\mathcal{L}_{2n+1}$:

$$\begin{aligned}\mathcal{L}_{2n+1} &= \frac{1}{2}\mathcal{L}_{2n} + \frac{9}{2}\mathcal{J}_{2n} \\ &= \frac{1}{2^{n+1}}\int_{-1}^{1} (5+3(n+1)x)\,(5+3x)^{n-1}\,dx \\ &\quad + \frac{9}{2}\frac{n}{2^n}\int_{-1}^{1} (5+3x)^{n-1}\,dx \\ &= \frac{1}{2^{n+1}}\int_{-1}^{1} (5+9n+3(n+1)x)\,(5+3x)^{n-1}\,dx.\end{aligned}$$

Thus, the proof of Corollary 3.7 is completed. □

The following corollary gives a thinly disguised form of Binet's formula for $\mathcal{L}_{kn}$.

Corollary 3.8. *The Jacobsthal-Lucas numbers $\mathcal{L}_{kn}$ can be represented by*

$$\mathcal{L}_{kn} = n\int_{(-1)^k}^{\alpha^k} t^{n-1}dt.$$

for $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$.

Proof. The proof of Corollary 3.8 can be done similarly to the proof of Corollary 3.4. □

In the following section, we obtain the integral representations for the Jacobsthal Numbers $\mathcal{J}_{kn+r}$ and the Jacobsthal-Lucas Numbers $\mathcal{L}_{kn+r}$. As a result of substituting $(1,0)$, $(2,0)$, and $(2,1)$ for (k,r), we are able to obtain integral representations for $\mathcal{J}_n$, $\mathcal{J}_{2n}$, $\mathcal{J}_{2n+1}$, $\mathcal{L}_n$, $\mathcal{L}_{2n}$ and $\mathcal{L}_{2n+1}$.

4. Integral representations for the Jacobsthal Numbers $\mathcal{J}_{kn+r}$ and the Jacobsthal-Lucas Numbers $\mathcal{L}_{kn+r}$

This section presents the integral representations of Jacobsthal numbers $\mathcal{J}_{kn+r}$ and Jacobsthal-Lucas numbers $\mathcal{L}_{kn+r}$, derived from the integral representations of Jacobsthal numbers $\mathcal{J}_{kn}$ and Jacobsthal-Lucas numbers $\mathcal{L}_{kn}$, where $n \in \mathbb{Z}_{\geqslant 0} = \{0,1,2,\ldots\}$ is a non-negative integer, $k \in \mathbb{Z}_{>0} = \{1,2,3,\ldots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer.

Theorem 4.1 presents the integral representations of Jacobsthal numbers $\mathcal{J}_{kn+r}$. Therefore, in the integral representation given by Theorem 4.1, substituting different integer pairs (k,r) yields integral representations for different Jacobsthal numbers $\mathcal{J}_{kn+r}$.

Theorem 4.1. *For $n \in \mathbb{Z}_{\geqslant 0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geqslant 0}$, the Jacobsthal numbers $\mathcal{J}_{kn+r}$ can be represented by the integral*

$$\mathcal{J}_{kn+r} = \frac{1}{2^{n+1}}\int_{-1}^{1} (n\mathcal{J}_k\mathcal{L}_r + \mathcal{J}_r\mathcal{L}_k + 3\mathcal{J}_k\mathcal{J}_r(n+1)x)\,(\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1}\,dx. \tag{22}$$

Proof. The Jacobsthal index addition formula

$$\mathcal{J}_r\mathcal{L}_m + \mathcal{L}_r\mathcal{J}_m = 2\mathcal{J}_{m+r}$$

given by (7) with m replaced with kn produces

$$2\mathcal{J}_{kn+r} = \mathcal{J}_{kn}\mathcal{L}_r + \mathcal{J}_r\mathcal{L}_{kn}.$$

With the help of formulas (9) and (15), we can express $\mathcal{J}_{kn}$ and $\mathcal{L}_{kn}$ in terms of $\mathcal{J}_k$, $\mathcal{L}_r$, $\mathcal{J}_r$ and $\mathcal{L}_k$. This allows us to obtain an integral representation for $\mathcal{J}_{kn+r}$. By substituting the integral representations of $\mathcal{J}_{kn}$ and $\mathcal{L}_{kn}$ given by (9) and (15) respectively, the following result emerges immediately:

$$\begin{aligned} 2\mathcal{J}_{kn+r} &= \mathcal{L}_r\mathcal{J}_{kn} + \mathcal{J}_r\mathcal{L}_{kn} \\ &= \mathcal{L}_r\frac{n\mathcal{J}_k}{2^n}\int_{-1}^{1}(\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1}\,dx \\ &\quad + \mathcal{J}_r\frac{1}{2^n}\int_{-1}^{1}(\mathcal{L}_k + 3\mathcal{J}_k(n+1)x)(\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1}\,dx \\ &= \frac{1}{2^n}\int_{-1}^{1}(n\mathcal{J}_k\mathcal{L}_r + \mathcal{J}_r\mathcal{L}_k + 3\mathcal{J}_k\mathcal{J}_r(n+1)x)(\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1}\,dx, \end{aligned}$$

and completes the proof. □

As indicated in the following remark, the results in Corollary 3.1, Corollary 3.2 and Corollary 3.3 can also be obtained using Theorem 4.1.

Remark 4.1. *In the integral representation at (22) given by Theorem 4.1, substituting* $(1,0)$, $(2,0)$, *and* $(2,1)$ *for* (k,r) *yields integral representations for* $\mathcal{J}_n$, $\mathcal{J}_{2n}$, *and* $\mathcal{J}_{2n+1}$.

Theorem 4.2 presents the integral representations of Jacobsthal-Lucas numbers $\mathcal{L}_{kn+r}$. Therefore, in the integral representation given by Theorem 4.2, substituting different integer pairs (k,r) yields integral representations for different Jacobsthal-Lucas numbers $\mathcal{L}_{kn+r}$.

Theorem 4.2. *For* $n \in \mathbb{Z}_{\geqslant 0}$ *and arbitrary but fixed* $k \in \mathbb{Z}_{>0}$ *and* $r \in \mathbb{Z}_{\geqslant 0}$, *the Jacobsthal-Lucas numbers* $\mathcal{L}_{kn+r}$ *can be represented by the integral*

$$\mathcal{L}_{kn+r} = \frac{1}{2^{n+1}}\int_{-1}^{1}(9n\mathcal{J}_k\mathcal{J}_r + \mathcal{L}_k\mathcal{L}_r + 3\mathcal{J}_k\mathcal{L}_r(n+1)x)(\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1}\,dx. \tag{23}$$

Proof. The Jacobsthal-Lucas index addition formula

$$\mathcal{L}_m\mathcal{L}_r + 9\mathcal{J}_m\mathcal{J}_r = 2\mathcal{L}_{m+r}$$

given by (1.6) with m replaced with kn produces

$$\mathcal{L}_{kn}\mathcal{L}_r + 9\mathcal{J}_{kn}\mathcal{J}_r = 2\mathcal{L}_{kn+r}.$$

Formulas (9) and (15) allow us to write $\mathcal{J}_{kn}$ and $\mathcal{L}_{kn}$ in terms of $\mathcal{J}_k$, $\mathcal{L}_r$, $\mathcal{J}_r$ and $\mathcal{L}_k$. In this way, we can obtain an integral representation for $\mathcal{J}_{kn+r}$. When the integral representations of $\mathcal{J}_{kn}$ and $\mathcal{L}_{kn}$ given by (9) and (15) respectively are substituted into the given index addition formula, the result follows immediately:

$$\begin{aligned}
2\mathcal{L}_{kn+r} &= 9\mathcal{J}_r\mathcal{J}_{kn} + \mathcal{L}_r\mathcal{L}_{kn} \\
&= 9\mathcal{J}_r\frac{n\mathcal{J}_k}{2^n}\int_{-1}^{1}(\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1}\,dx \\
&\quad + \mathcal{L}_r\frac{1}{2^n}\int_{-1}^{1}(\mathcal{L}_k + 3\mathcal{J}_k(n+1)x)(\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1}\,dx \\
&= \frac{1}{2^n}\int_{-1}^{1}(9n\mathcal{J}_k\mathcal{J}_r + \mathcal{L}_k\mathcal{L}_r + 3\mathcal{J}_k\mathcal{L}_r(n+1)x)(\mathcal{L}_k + 3\mathcal{J}_k x)^{n-1}\,dx,
\end{aligned}$$

and completes the proof. □

In the following remark, it is indicated that the results given in Corollary 3.5, Corollary 3.6 and Corollary 3.7 can be obtained by using Thereom 4.2.

Remark 4.2. *In the integral representation at (23) given by Theorem 4.2, substituting* $(1,0)$, $(2,0)$, *and* $(2,1)$ *for* (k,r) *yields integral representations for* $\mathcal{L}_n$, $\mathcal{L}_{2n}$ *and* $\mathcal{L}_{2n+1}$.

5. Conclusion

In the first part of this note, integral representations are obtained for Jacobsthal numbers $\mathcal{J}_{kn}$ as well as Jacobsthal-Lucas numbers $\mathcal{L}_{kn}$, and then based on those integral representations, integral representations are given for Jacobsthal numbers $\mathcal{J}_{kn+r}$ and Jacobsthal-Lucas numbers $\mathcal{L}_{kn+r}$, where $n \in \mathbb{Z}_{\geqslant 0} = \{0,1,2,\ldots\}$ is a non-negative integer, $k \in \mathbb{Z}_{>0} = \{1,2,3,\ldots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer.

Acknowledgment

The author would like to thank the Editor and the referees for their very helpful comments and suggestions.

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