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# Type II half logistic Lomax-Weibull distribution: properties, simulations and applications

**S. A. Osagie[1], F. C. Opone[2*] and J. E. Osemwenkhae[3]**
[1,3]Department of Statistics, University of Benin, Benin City, Nigeria.
[2]Department of Statistics, Delta State University of Science and Technology, Ozoro, Nigeria.

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**Abstract:** The relevance of lifetime distributions in the theory of statistical modeling has attracted a commendable effort from researchers to continue developing more tractable and flexible lifetime distributions to tackle the unfolding complex data in various fields of study. In this paper, a new five-parameter lifetime distribution is developed by utilizing the Lomax-Weibull distribution as a baseline distribution in the well-known Type II Half Logistic-G family. The resulting distribution is known as the Type II Half Logistic Lomax-Weibull (TIIHLLW) distribution. The TIIHLLW distribution has the advantage of modeling data sets exhibiting monotonic and non-monotonic failure rates. A comprehensive investigation of several statistical properties of the TIIHLLW distribution is carried out. The use of the maximum likelihood estimation method to obtain parameter estimates of the TIIHLLW distribution is discussed and a Monte Carlo simulation experiment is performed to examine the asymptotic behavior of the maximum likelihood estimators of the TIIHLLW distribution premised on their means, biases, and root mean square errors. The versatility and flexibility of the TIIHLLW distribution are illustrated via two real lifetime data sets. Summary results from the fitting of the two data sets revealed that the TIIHLLW distribution provided a reasonably better fit than some existing competing distributions.

**Key words:** T-X method, Probability Weighted Moments, Entropy, Simulation study, Mean deviations

## 1. Introduction

In the last two decades, several lifetime distributions have been developed from classical ones to model complex data sets obtained from real-life problems. An immediate purpose for construction of these generalized distributions is to enhance the flexibility and widen the scope of application of the classical distribution. [15] and [14] discussed the different methods of generating distributions and families of distributions using continuous univariate distributions. Notable among these methods of generalization include the exponentiated-G family distributions due to [18], Marshall-Olkin family of distributions developed by [17], the transmuted family of distributions studied by [26], the beta generated family of distributions proposed by [11]. Another tractable method explored in developing families of lifetime distributions is based on the transformed-transformer ($T$-$X$) method proposed by [15]. The cumulative distribution function (cdf) of the $T$-$X$ class is defined by

$$G_{T-X}(x) = \int_{c}^{W[F(x)]} r(t)dt = R(W[F(x)]), \tag{1}$$

Let $r(t)$ and $R(t)$ define the pdf and cdf of a random variable $T \in (c, d)$ for $-\infty \leq c < d < \infty$ and let $W[F(x)]$ be a function of $f(x)$ of random variable $X$ satisfying the conditions,


*Correspondence: oponef@dsust.edu.ng

1. $W[F(x)] \in [c, d]$,
2. $W[F(x)]$ is differentiable and monotonically increasing, and
3. $W[F(x)] \to c$ as $x \to -\infty$ and $W[F(x)] \to d$ as $x \to \infty$.

Utilizing the technique in (1), several other families of lifetime distributions have been developed in the literature. For instance, [28] proposed the new Kumaraswamy-G family, [3] introduced the Weibull-$X$ family of distributions, and [6] introduced the odd Chen-G family. Other generalized families include the Type II half logistic-G family developed by [27], Type II half logistic odd Fréchet-G family studied in [4] and odd Lomax generator proposed by [9], the continuous Bernoulli generated family of distributions developed by [30], the Alpha Power Type II Topp-Leone family of distributions introduced by [10], and the Opone-generated family of distributions proposed by [22] . These generalized methods of constructing new families of distributions have been adapted by researchers to extend classical distributions.

A notable family of distributions is the Type II half logistic-G (TIIHL-G) family of distributions due to [27], with the cdf given as

$$G_{TIIHL-G}(x) = \frac{2[Z(x)]^{\eta}}{1 + [Z(x)]^{\eta}}. \tag{2}$$

Using (1), the authors defined $W[F(x)] = -log_e Z(x)$ and $r(t)$ to be the pdf of the logistic distribution defined by $\frac{2\lambda exp^{-\lambda t}}{(1+exp^{-\lambda t})^2}$, $t > 0$.

The direct aim of this paper is to increase the flexibility and applicability of the Lomax-Weibull distribution introduced by [23]. This is achieved by utilizing the density function of the Lomax-Weibull distribution as the baseline distribution in the framework defined in (2). The paper has been organized into the following sections. Section 2 considers the formulation of the TIIHLLW distribution and an extensive study of the general statistical properties. Section 3 presents a Monte Carlo simulation experiment and parameter estimation of the TIIHLLW distribution is considered. Section 4 presents real-life data fittings of TIIHLLW distribution alongside some nested and non-nested distributions. Finally, Section 5 provides the concluding remark.

## 2. The TIIHLLW Distribution

The four-parameter Lomax-Weibull (LW) distribution was developed by [23], with the cdf specified as

$$Z(x) = 1 - (1 + \theta x)^{-\lambda} e^{-\gamma x^{\beta}}; \qquad x > 0, \theta > 0, \lambda > 0, \gamma > 0, \beta > 0. \tag{3}$$

Substituting (3) as baseline distribution into (2), the cdf of the five-parameter TIIHLLW distribution is obtained as

$$G(x) = \frac{2[1 - (1 + \theta x)^{-\lambda} e^{-\gamma x^{\beta}}]^{\eta}}{1 + [1 - (1 + \theta x)^{-\lambda} e^{-\gamma x^{\beta}}]^{\eta}}. \tag{4}$$

The corresponding probability density function (pdf), survival, and hazard functions are, respectively, given as

$$g(x) = \frac{2\eta(\lambda\theta + \beta\gamma x^{\beta-1}(1 + \theta x))(1 + \theta x)^{-\lambda-1} e^{-\gamma x^{\beta}} [1 - (1 + \theta x)^{-\lambda} e^{-\gamma x^{\beta}}]^{\eta-1}}{[1 + [1 - (1 + \theta x)^{-\lambda} e^{-\gamma x^{\beta}}]^{\eta}]^2}, \tag{5}$$

$$\bar{G}(x) = \frac{1 - \left[1 - (1 + \theta x)^{-\lambda} e^{-\gamma x^{\beta}}\right]^{\eta}}{1 + \left[1 - (1 + \theta x)^{-\lambda} e^{-\gamma x^{\beta}}\right]^{\eta}}, \tag{6}$$

and

$$h(x) = \frac{2\eta(\lambda\theta + \beta\gamma x^{\beta-1}(1+\theta x))(1+\theta x)^{-\lambda-1}e^{-\gamma x^{\beta}}[1-(1+\theta x)^{-\lambda}e^{-\gamma x^{\beta}}]^{\eta-1}}{1-[1-(1+\theta x)^{-\lambda}e^{-\gamma x^{\beta}}]^{2\eta}}. \tag{7}$$

Figure 1 presents plots illustrating the different shapes of the pdf and hazard function of the TIIHLLW distribution with varying values of the parameters.

**Figure 1**. Different shapes of (a) TIIHLLW g(x) and (b) TIIHLLW h(x)

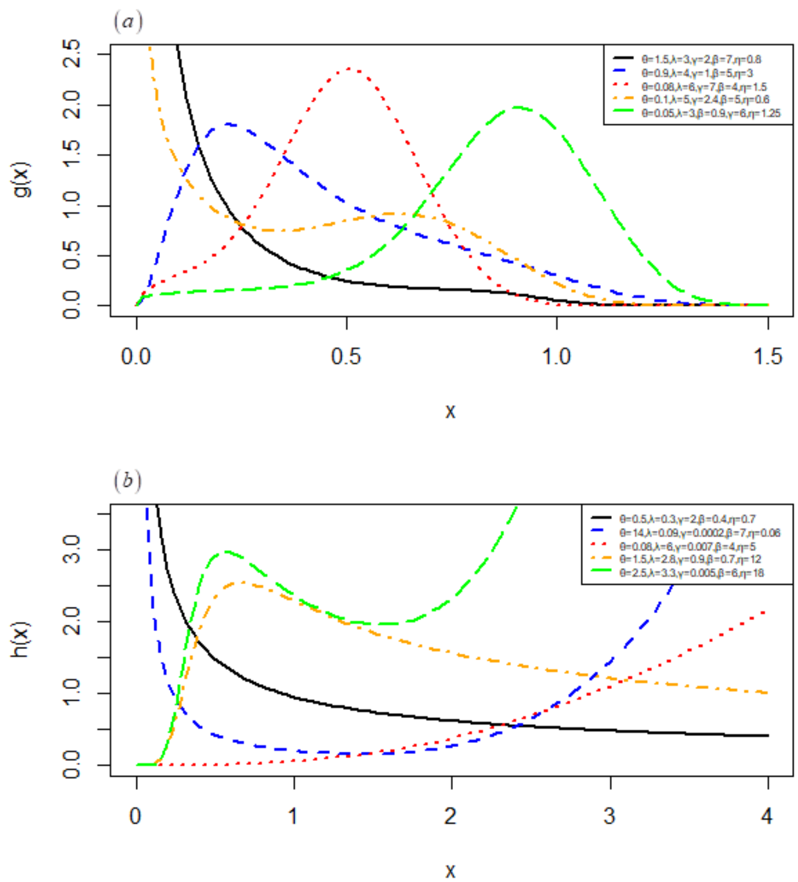



Figure 1(a) simply suggests that the density function of the TIIHLLW distribution accommodates decreasing, left-skewed, right-skewed, or approximately symmetric shapes. Whereas, Figure 1(b) indicates that the TIIHLLW distribution exhibits decreasing, bathtub, and inverted bathtub shapes which are useful in fitting data sets with monotonic and non-monotonic failure rates.

### 2.1. Linear Representation

In this subsection, the series expansion of the probability density function (pdf) in (5) is obtained. This action reduces computational stress and allows explicit expression for other structural properties characterizing the TIIHLLW distribution.

The following binomial expansions are useful in achieving this task.

$$(1+az)^n = \sum_{i=0}^{n} \binom{n}{i} a^i z^i,$$

and

$$(1+az)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} (-1)^i a^i z^i.$$

Now, the series expansion of the TIIHLLW pdf is derived as follows:

$$g(x) = 2\eta(\lambda\theta + \beta\gamma x^{\beta-1}(1+\theta x))(1+\theta x)^{-\lambda-1} e^{-\gamma x^{\beta}} \sum_{i=0}^{\infty} \binom{1+i}{i} (-1)^i (1-(1+\theta x)^{-\lambda} e^{-\gamma x^{\beta}})^{\eta(i+1)-1}.$$

Further expansion gives

$$g(x) = 2\eta(\lambda\theta + \beta\gamma x^{\beta-1}(1+\theta x)) \sum_{i,j=0}^{\infty} \binom{1+i}{i} \binom{\eta(1+i)-1}{j} (-1)^{i+j} (1+\theta x)^{-\lambda(j+1)-1} e^{-\gamma(j+1)x^{\beta}},$$

$$= 2\eta(\lambda\theta + \beta\gamma x^{\beta-1}(1+\theta x)) \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} x^{k\beta} (1+\theta x)^{-\lambda(j+1)-1},$$

where $\nu_{i,j,k} = \binom{1+i}{i} \binom{\eta(1+i)-1}{j} \frac{(-1)^{i+j+k}(\gamma(j+1))^k}{k!}$.

Hence,

$$g(x) = 2\eta \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} (1+\theta x)^{-\lambda(j+1)-1} (\lambda\theta x^{k\beta} + \beta\gamma x^{(k+1)\beta-1}(1+\theta x)). \tag{8}$$

### 2.2. Quantile function

For $0 < u < 1$, the quantile function of the TIIHLLW distribution can be obtained using the expression

$$G(X_u) = u.$$

which gives

$$\frac{2[1-(1+\theta X_u)^{-\lambda} e^{-\gamma X_u^{\beta}}]^{\eta}}{1+[1-(1+\theta X_u)^{-\lambda} e^{-\gamma X_u^{\beta}}]^{\eta}} = u. \tag{9}$$

After several algebraic simplifications, the quantile function of the TIIHLLW distribution is obtained as

$$\lambda ln\Big(1+\theta X_u\Big) + \gamma X_u^{\beta} + ln\left(1 - \left(\frac{u}{2-u}\right)^{\frac{1}{\eta}}\right) = 0. \tag{10}$$

An explicit solution does not exist for the non-linear equation in (10) regarding the parameters of interest. Hence, the root of the non-linear equation in (10) will be obtained by any known iterative method for selected parameter values. The numerical values of the TIIHLLW quantile function for some selected parameter values of $\theta, \lambda, \gamma, \beta, \eta$ are presented in Table 1.

From Table 1, the quantile function of the TIIHLLW distribution exhibits a monotonically increasing nature for the selected values of $\theta, \lambda, \gamma, \beta, \eta$.

**Table 1**. Numerical values for TIIHLLW quantile function

| q | (8,0.7,1.3,2.5,4) | (0.6,0.4,0.5,0.5,12) | (7,0.9,0.5,1,2) | (0.7,6.3,1.5,8,15) | (0.05,7,2,0.1,9) |
|---|---|---|---|---|---|
| 0.1 | 0.1838237 | 4.184694 | 0.04340951 | 0.4491119 | 0.01087875 |
| 0.2 | 0.2729501 | 5.697963 | 0.07246225 | 0.5287879 | 0.05968824 |
| 0.3 | 0.3573622 | 7.166087 | 0.10416448 | 0.5942134 | 0.17515587 |
| 0.4 | 0.4423029 | 8.777901 | 0.14227861 | 0.6546412 | 0.37917981 |
| 0.5 | 0.5304991 | 10.69145 | 0.19154221 | 0.7128879 | 0.68840934 |
| 0.6 | 0.6251269 | 13.13529 | 0.26005390 | 0.7700785 | 1.13059954 |
| 0.7 | 0.7315213 | 16.54384 | 0.36477546 | 0.8274321 | 1.76973251 |
| 0.8 | 0.8608155 | 21.97110 | 0.54968911 | 0.8878493 | 2.76972480 |
| 0.9 | 1.0451345 | 33.23814 | 0.98257826 | 0.9600337 | 4.70660275 |

### 2.3. Raw moments

For random variable $X$ following any continuous distribution, the $r^{th}$ raw moments are defined as

$$\mu^r = E(X^r) = \int_0^\infty x^r g(x)dx.$$

Substituting (8) into the above expression, we have

$$\mu^r = 2\eta \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \left( \lambda\theta \int_0^\infty x^{r+k\beta}(1+\theta x)^{-\lambda(j+1)-1}dx \right.$$
$$\left. +\beta\gamma \int_0^\infty x^{r+(k+1)\beta-1}(1+\theta x)^{-\lambda(j+1)}dx \right). \tag{11}$$

Using the integral solution given as

$$\int_0^\infty \frac{x^{a-1}}{(1+bx)^c}dx = \frac{B(a, c-a)}{b^a},$$

where $B(m, n)$ is the beta function.
The $r^{th}$ raw moments of the TIIHLLW distribution is, thus, given as

$$\mu^r = 2\eta \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \left( \frac{\lambda B(r+k\beta+1, \lambda(j+1)-(r+k\beta))}{\theta^{r+k\beta}} + \frac{\beta\gamma B(r+(k+1)\beta, \lambda(j+1)-(r+(k+1)\beta))}{\theta^{r+(k+1)\beta}} \right). \tag{12}$$

Computations for standard deviation (SD), coefficient of variations (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) can be obtained from (12) using the following expressions.

$$SD = E(X^2) - \mu^2; \qquad CV = \frac{Var(X)}{\mu}; \qquad CS = \frac{\left(E(X^3) - 3\mu E(X^2) + 2\mu^3\right)^2}{\left(E(X^2) - \mu^2\right)^3};$$
$$CK = \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{\left(E(X^2) - \mu^2\right)^2}.$$

Table 2 presents results for the first four $r^{th}$ raw moments and other related quantities for TIIHLLW distribution using sets of arbitrary parameter values.

**Table 2**. Raw moments and related quantities for TIIHLLW distribution

| Quantities | (4.0,0.1,0.5,2.7,1.0) | (0.9,3.0,0.6,7.0,0.08) | (0.2,0.5,0.9,0.7,0.6) | (0.06,4.5,0.5,1.7,9.0) | (3.0,4.0,0.6,0.8,0.05) |
|---|---|---|---|---|---|
| $E(X)$ | 0.11285 | 0.02636 | 0.50467 | 2.10858 | 0.00377 |
| $E(X^2)$ | 0.10687 | 0.01397 | 1.61138 | 4.81874 | 0.00075 |
| $E(X^3)$ | 0.13296 | 0.01096 | 11.1756 | 11.9405 | 0.00043 |
| $E(X^4)$ | 0.19201 | 0.00996 | 126.339 | 32.0833 | 0.00055 |
| SD | 0.30681 | 10.11520 | 1.16477 | 0.61009 | 0.02723 |
| CV | 2.71881 | 4.37002 | 2.30799 | 0.28932 | 7.22202 |
| CS | 11.9039 | 41.8719 | 32.3868 | 0.85184 | 429.638 |
| CK | 22.5733 | 56.6318 | 64.7413 | 82.1057 | 1004.43 |

### 2.4. Incomplete moments

For random variable $X$ following any continuous distribution, the $r^{th}$ raw incomplete moments are defined as

$$\mu_t^r = E(X^r/X > t) = \frac{1}{\bar{G}(t)} \int_t^\infty x^r g(x) dx.$$

Substituting (6) and (8) into the above expression,

$$\mu_t^r = 2\eta \left( \frac{1 + [1 - (1+\theta x)^{-\lambda} e^{-\gamma x^\beta}]^\eta}{1 - [1 - (1+\theta x)^{-\lambda} e^{-\gamma x^\beta}]^\eta} \right) \sum_{i,j,k=0}^\infty \nu_{i,j,k} \left( \lambda\theta \int_t^\infty x^{r+k\beta} (1+\theta x)^{-\lambda(j+1)-1} dx \right.$$
$$\left. + \beta\gamma \int_t^\infty x^{r+(k+1)\beta-1} (1+\theta x)^{-\lambda(j+1)} dx \right). \tag{13}$$

If $z = (1+\theta x)^{-1}$, then

$$\mu_t^r = 2\eta \left( \frac{1 + [1 - (1+\theta x)^{-\lambda} e^{-\gamma x^\beta}]^\eta}{1 - [1 - (1+\theta x)^{-\lambda} e^{-\gamma x^\beta}]^\eta} \right) \sum_{i,j,k=0}^\infty \binom{1+i}{i} \binom{\eta(1+i)-1}{j}$$
$$\times \frac{(-1)^{i+j+k+1} (\gamma(j+1))^k}{k!} \left( \frac{\lambda}{\theta^{r+k\beta}} \int_0^{(1+\theta t)^{-1}} z^{\lambda(j+1)-r-k\beta-1} (1-z)^{r+k\beta} dz \right.$$
$$\left. + \frac{\beta\gamma}{\theta^{r+(k+1)\beta}} \int_0^{(1+\theta t)^{-1}} z^{\lambda(j+1)-r-(k+1)\beta-1} (1-z)^{r+(k+1)\beta-1} dz \right).$$

The $r^{th}$ raw incomplete moments of the TIIHLLW distribution are given as

$$\mu_t^r = 2\eta \left( \frac{1 + [1 - (1+\theta x)^{-\lambda} e^{-\gamma x^\beta}]^\eta}{1 - [1 - (1+\theta x)^{-\lambda} e^{-\gamma x^\beta}]^\eta} \right) \sum_{i,j,k=0}^\infty \binom{1+i}{i} \binom{\eta(1+i)-1}{j}$$
$$\times \frac{(-1)^{i+j+k+1} (\gamma(j+1))^k}{k!} \left[ \frac{\lambda}{\theta^{r+k\beta}} B_{(1+\theta t)^{-1}}(\lambda(j+1) - r - k\beta, r + k\beta + 1), \right. \tag{14}$$
$$\left. + \frac{\beta\gamma}{\theta^{r+(k+1)\beta}} B_{(1+\theta t)^{-1}}(\lambda(j+1) - r - (k+1)\beta, r + (k+1)\beta) \right],$$

where $B_y(u,v) = \int_0^y z^{u-1}(1-z)^{v-1}$ is the lower incomplete beta function.

### 2.5. Moment generating function

For a random variable $X$ following any continuous distribution, the moment generating function (mgf) denoted by $M_X(t)$ is defined as

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} g(x) dx.$$

Substituting (8) into the above expression, we have

$$M_X(t) = 2\eta \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \left( \lambda\theta \int_0^\infty x^{k\beta} e^{tx} (1+\theta x)^{-\lambda(j+1)-1} dx + \beta\gamma \int_0^\infty x^{(k+1)\beta-1} e^{tx} (1+\theta x)^{-\lambda(j+1)} dx \right).$$

Further simplification yields

$$M_X(t) = 2\eta \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \left[ \sum_{l=0}^{\infty} \frac{t^l}{l!} \left( \lambda\theta \int_0^\infty x^{l+k\beta} (1+\theta x)^{-\lambda(j+1)-1} dx + \beta\gamma \int_0^\infty x^{l+(k+1)\beta-1} (1+\theta x)^{-\lambda(j+1)} dx \right) \right].$$

Hence, the mgf of TIIHLLW distribution is expressed as

$$M_X(t) = 2\eta \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \Big[ \sum_{l=0}^{\infty} \frac{t^l}{l!} \Big( \frac{\lambda B(l+k\beta+1, \lambda(j+1)-(l+k\beta))}{\theta^{r+k\beta}} + \frac{\beta\gamma B(l+(k+1)\beta, \lambda(j+1)-(l+(k+1)\beta))}{\theta^{l+(k+1)\beta}} \Big) \Big].$$

### 2.6. Probability weighted moments

The probability weighted moment ($PWM$) of a random variable $X$ following any continuous distribution is defined as

$$PWM_{r,l,m}(X) = E(X^r G^l(X) \bar{G}^m(X)) = \int_0^\infty x^r G^l(x) \bar{G}^m(x) g(x) dx,$$
$$= \int_0^\infty x^r G^l(x) (1-G(x))^m g(x) dx = \sum_{i=0}^{m} \binom{m}{w} (-1)^m \int_0^\infty x^r G^{l+w}(x) g(x) dx.$$

Substituting (4) and (5) into the above expression, the $PWM$ of TIIHLLW distribution can be obtained as

$$PWM_{r,l,m}(X) = \sum_{i=0}^{\infty} \binom{w}{m} (-1)^m 2^{l+w+1} \eta \int_0^\infty x^r (\lambda\theta + \beta\gamma x^{\beta-1}(1+\theta x))$$
$$\times (1+\theta x)^{-\lambda-1} e^{-\gamma x^\beta} [1-(1+\theta x)^{-\lambda} e^{-\gamma x^\beta}]^{\eta-1} \left( \frac{2[1-(1+\theta x)^{-\lambda} e^{-\gamma x^\beta}]^\eta}{1+[1-(1+\theta x)^{-\lambda} e^{-\gamma x^\beta}]^\eta} \right)^{l+w} dx.$$

After further algebraic expansions, the $PWM$ of TIIHLLW distribution is expressed as

$$PWM_{r,l,m}(X) = \eta \sum_{w,i,j,k=0}^{\infty} \binom{m}{w} \binom{1+l+w+i}{i} \binom{\eta(1+l+w+i)-1}{j}$$
$$\times \frac{2^{l+w+1}(-1)^{w+i+j+k}(\gamma(j+1))^k}{k!} \left( \frac{\lambda B(r+k\beta+1, \lambda(j+1)-(r+k\beta))}{\theta^{r+k\beta}} \right.$$
$$\left. + \frac{\beta\gamma B(r+(k+1)\beta, \lambda(j+1)-(r+(k+1)\beta))}{\theta^{r+(k+1)\beta}} \right). \tag{15}$$

### 2.7. Mean deviations

Mean deviation (MD) is a measure of the variations or the amount of scatter in any data set from the mean and median. The mean deviations about the mean $(\mu)$ and median $(M)$ for any random variable $X$ are defined as

$$MD_\mu = E(|X-\mu|) = \int_0^\mu |x-\mu| g(x)dx = 2\mu G(\mu) - 2\int_0^\mu x g(x)dx,$$
$$MD_M = E(|X-M|) = \int_0^M |x-M| g(x)dx = \mu - 2\int_0^M x g(x)dx.$$

Substituting (8) into the above expression, the mean deviations of the TIIHLLW distribution becomes

$$MD_\mu = \frac{4\mu[1-(1+\theta\mu)^{-\lambda}e^{-\gamma\mu^\beta}]^\eta}{1+[1-(1+\theta\mu)^{-\lambda}e^{-\gamma\mu^\beta}]^\eta} - 4\eta \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \left( \lambda\theta \int_0^\mu x^{1+k\beta}(1+\theta x)^{-\lambda(j+1)-1}dx \right.$$
$$\left. + \beta\gamma \int_0^\mu x^{1+(k+1)\beta-1}(1+\theta x)^{-\lambda(j+1)}dx \right),$$

and

$$MD_M = \mu - 4\eta \sum_{i,j,k=0}^{\infty} \nu_{i,j,k} \left( \lambda\theta \int_0^\mu x^{1+k\beta}(1+\theta x)^{-\lambda(j+1)-1}dx + \beta\gamma \int_0^\mu x^{1+(k+1)\beta-1}(1+\theta x)^{-\lambda(j+1)}dx \right).$$

Hence, the mean deviations of the TIIHLLW distribution are defined as

$$MD_\mu = \frac{4\mu[1-(1+\theta\mu)^{-\lambda}e^{-\gamma\mu^\beta}]^\eta}{1+[1-(1+\theta\mu)^{-\lambda}e^{-\gamma\mu^\beta}]^\eta} - 4\eta \sum_{i,j,k=0}^{\infty} \binom{1+i}{i} \binom{\eta(1+i)-1}{j}$$
$$\times \frac{(-1)^{i+j+k+1}(\gamma(j+1))^k}{k!} \left[ \frac{\lambda}{\theta^{1+k\beta}} \left( B_{(1+\theta\mu)^{-1}}(\lambda(j+1)-1-k\beta, 2+k\beta) \right) \right.$$
$$\left. + \frac{\beta\gamma}{\theta^{1+(k+1)\beta}} \left( B_{(1+\theta\mu)^{-1}}(\lambda(j+1)-1-(k+1)\beta, 1+(k+1)\beta) \right) \right],$$

and

$$MD_M = \mu - 4\eta \sum_{i,j,k=0}^{\infty} \binom{1+i}{i} \binom{\eta(1+i)-1}{j} \frac{(-1)^{i+j+k+1}(\gamma(j+1))^k}{k!}$$
$$\times \left[ \frac{\lambda}{\theta^{1+k\beta}} \left( B_{(1+\theta M)^{-1}}(\lambda(j+1)-1-k\beta, 2+k\beta) \right) \right.$$
$$\left. + \frac{\beta\gamma}{\theta^{1+(k+1)\beta}} \left( B_{(1+\theta M)^{-1}}(\lambda(j+1)-1-(k+1)\beta, 1+(k+1)\beta) \right) \right].$$

### 2.8. Distribution of order statistic

Suppose $X_{(i)}, i = 1(1)n$ is the $i^{th}$ order statistic such that $X_1, X_2, ..., X_n$ define a random sample of $n$ variables. Then the pdf of $i^{th}$ order statistic for TIIHLLW distribution can be obtained by

$$g_{i,j}(x) = \frac{j!}{(i-1)!(j-i)!} \left[1 - \bar{G}(x)\right]^{i-1} \left[\bar{G}(x)\right]^{j-i} g(x),$$
$$= \frac{j!}{(i-1)!(j-i)!} \sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^k \left[\bar{G}(x)\right]^{j+k-i} g(x).$$

Substituting (5) and (6) into the above expression, the pdf of the order statistics of TIIHLLW distribution is given as

$$g_{i,j}(x) = \frac{j!}{(i-1)!(j-i)!} \sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^k \left[ \frac{1-[1-(1+\theta x)^{-\lambda} e^{-\gamma x^{\beta}}]^{\eta}}{1+[1-(1+\theta x)^{-\lambda} e^{-\gamma x^{\beta}}]^{\eta}} \right]^{j+k-i}$$
$$\times \frac{2\eta(\lambda\theta + \beta\gamma x^{\beta-1}(1+\theta x))(1+\theta x)^{-\lambda-1} e^{-\gamma x^{\beta}} [1-(1+\theta x)^{-\lambda} e^{-\gamma x^{\beta}}]^{\eta-1}}{[1+[1-(1+\theta x)^{-\lambda} e^{-\gamma x^{\beta}}]^{\eta}]^2},$$

$$= \frac{j!2\eta}{(i-1)!(j-i)!} \sum_{k=0}^{i-1} \sum_{\nu=0}^{j+k-i} \sum_{\omega,\zeta,\rho=0}^{\infty} \binom{i-1}{k} \binom{1+j+k-i+\omega}{\omega} \binom{\eta(\nu+\omega+1)-1}{\zeta}$$
$$\times \frac{(-1)^{k+\nu+\omega+\zeta}((\zeta+1)\gamma)^{\rho}}{\rho!} x^{\rho\beta}(\lambda\theta + \beta\gamma x^{\beta-1}(1+\theta x))(1+\theta x)^{-\lambda(\zeta+1)-1}. \quad (16)$$

The corresponding $r^{th}$ raw moments of order statistic of TIIHLLW distribution can be obtained as follows.

$$\mu_{i,j}^{r} = \int_0^{\infty} x^r g_{i,j}(x) dx,$$

$$\mu_{i,j}^{r} = \frac{j!2\eta}{(i-1)!(j-i)!}\sum_{k=0}^{i-1}\sum_{\nu=0}^{j+k-i}\sum_{\omega,\zeta,\rho=0}^{\infty}\binom{i-1}{k}\binom{1+j+k-i+\omega}{\omega}\binom{\eta(\nu+\omega+1)-1}{\zeta}$$
$$\times\frac{(-1)^{k+\nu+\omega+\zeta}((\zeta+1)\gamma)^{\rho}}{\rho!}\int_{0}^{\infty}x^{r+\rho\beta}(\lambda\theta+\beta\gamma x^{\beta-1}(1+\theta x))(1+\theta x)^{-\lambda(\zeta+1)-1}dx,$$
$$= \frac{j!2\eta}{(i-1)!(j-i)!}\sum_{k=0}^{i-1}\sum_{\nu=0}^{j+k-i}\sum_{\omega,\zeta,\rho=0}^{\infty}\binom{i-1}{k}\binom{1+j+k-i+\omega}{\omega}\binom{\eta(\nu+\omega+1)-1}{\zeta} \quad (17)$$
$$\times\frac{(-1)^{k+\nu+\omega+\zeta}((\zeta+1)\gamma)^{\rho}}{\rho!}\left[\frac{\lambda B(r+\rho\beta+1,\lambda(\zeta+1)-r-\rho\beta)}{\theta^{r+\rho\beta}}\right.$$
$$\left.+\frac{\beta\gamma B(r+(\rho+1)\beta,\lambda(\zeta+1)-r-(\rho+1)\beta)}{\theta^{r+(\rho+1)\beta}}\right].$$

### 2.9. Residual lifetime

Residual lifetime measures the time remaining for a system beyond age $t$ until failure is known. It defines conditional random variable $X_t = [(X-t)/X>t)]$, $t \geq 0$. At the same time, the reversed residual lifetime (known as the inactivity lifetime) measures the elapsed lifetime for a system from its failure for which its lifetime is less than or equals to age $t \geq 0$. It defines conditional random variable $X_t^* = [(t-X)/X \leq t)]$, $t \geq 0$.
Suppose $m_r(t)$ and $m_r^*(t)$, $r = 1,2,3,...$ define the expectations of the residual and reversed residual lifetimes and are denoted as $E\left[(X-t)^r/X>t)\right]$ and $E\left[(t-X)^r/X \leq t)\right]$ denote the $r^{th}$ moments for residual and reversed residual lifetimes respectively, then the residual lifetime for a random variable following the TIIHLLW distribution is given as

$$m_r(t) = \frac{1}{\bar{G}(t)}\int_{t}^{\infty}(x-t)^r g(x)dx, \quad (18)$$

$$m_r(t) = \frac{2\eta(1+\left[1-(1+\theta x)^{-\lambda}e^{-\gamma x^{\beta}}\right]^{\eta})}{1-\left[1-(1+\theta x)^{-\lambda}e^{-\gamma x^{\beta}}\right]^{\eta}}\sum_{i,j,k=0}^{\infty}\nu_{i,j,k}\int_{t}^{\infty}(x-t)^r(1+\theta x)^{-\lambda(j+1)-1}$$
$$\times(1+\theta x)^{-\lambda(j+1)-1)}(\lambda\theta x^{k\beta}+\beta\gamma x^{(k+1)\beta-1}(1+\theta x))dx,$$

$$= \frac{2\eta(1+\left[1-(1+\theta x)^{-\lambda}e^{-\gamma x^{\beta}}\right]^{\eta})}{1-\left[1-(1+\theta x)^{-\lambda}e^{-\gamma x^{\beta}}\right]^{\eta}}\sum_{i,j,k=0}^{\infty}\nu_{i,j,k}\left\{\sum_{\omega=0}^{r}\binom{\omega}{r}(-1)^{\omega}t^{\omega}\right.$$
$$\left.\times\left[\lambda\theta\int_{t}^{\infty}x^{r+k\beta-\omega}(1+\theta x)^{-\lambda(j+1)-1}dx+\beta\gamma\int_{t}^{\infty}x^{r+(k+1)\beta-\omega-1}(1+\theta x)^{-\lambda(j+1)}dx\right]\right\}.$$

Hence, the expected value of the residual lifetime of the TIIHLLW distribution is expressed as

$$m_r(t) = \frac{2\eta(1+\left[1-(1+\theta x)^{-\lambda}e^{-\gamma x^{\beta}}\right]^{\eta})}{1-\left[1-(1+\theta x)^{-\lambda}e^{-\gamma x^{\beta}}\right]^{\eta}} \sum_{i,j,k=0}^{\infty} \nu_{i,j,k}\Bigg\{\sum_{\omega=0}^{r}\binom{\omega}{r}(-1)^{\omega}t^{\omega}$$
$$\times\Bigg[\frac{\lambda}{\theta^{r+k\beta}}B_{(1+\theta t)^{-1}}(\lambda(j+1)+\omega-r-k\beta, r+k\beta-\omega+1)$$
$$+\frac{\beta\gamma}{\theta^{r+(k+1)\beta}}B_{(1+\theta t)^{-1}}(\lambda(j+1)+\omega-r-(k+1)\beta, r+(k+1)\beta-\omega)\Bigg]\Bigg\}.$$

Also,

$$\begin{aligned} m_r^*(t) &= \frac{1}{G(t)}\int_0^t (t-x)^r g(x)dx, \\ &= \frac{\eta(1+[1-(1+\theta t)^{-\lambda}e^{-\gamma t^{\beta}}]^{\eta})}{[1-(1+\theta t)^{-\lambda}e^{-\gamma t^{\beta}}]^{\eta}} \sum_{i,j,k=0}^{\infty}\nu_{i,j,k}\Bigg\{\sum_{\omega=0}^{r}\binom{\omega}{r}(-1)^{r-\omega}t^{\omega} \\ &\times\Bigg[\lambda\theta\int_0^t x^{r+k\beta-\omega}(1+\theta x)^{-\lambda(j+1)-1}dx+\beta\gamma\int_0^t x^{r+(k+1)\beta-\omega-1}(1+\theta x)^{-\lambda(j+1)}dx\Bigg]\Bigg\}. \end{aligned}$$

Hence, the expected value of the reversed residual lifetime of the TIIHLLW distribution is expressed as

$$m_r^*(t) = \frac{\eta(1+[1-(1+\theta t)^{-\lambda}e^{-\gamma t^{\beta}}]^{\eta})}{[1-(1+\theta t)^{-\lambda}e^{-\gamma t^{\beta}}]^{\eta}} \sum_{i,j,k=0}^{\infty}\nu_{i,j,k}\Bigg[\sum_{\omega=0}^{r}\binom{\omega}{r}(-1)^{r-\omega}t^{\omega}$$
$$\times\Bigg(\frac{\lambda}{\theta^{r+k\beta}}B_{(1+\theta t)^{-1}}(\lambda(j+1)+\omega-r-k\beta, r+k\beta-\omega+1)$$
$$+\frac{\beta\gamma}{\theta^{r+(k+1)\beta}}B_{(1+\theta t)^{-1}}(\lambda(j+1)+\omega-r-(k+1)\beta, r+(k+1)\beta-\omega)\Bigg)\Bigg].$$

### 2.10. Entropy measures for TIIHLLW distribution

The R$\acute{e}$nyi and Shannon entropies (see [24] and [25]) are measures used in obtaining information about the randomness of random variables with applications found in many scientific fields for which lifetime analysis is applicable.

The R$\acute{e}$nyi entropy for the TIIHLLW distribution is obtained using

$$I_R(\rho) = \frac{1}{(1-\rho)}log\left(\int_0^{\infty} g^{\rho}(x)dx\right); \rho>0, \rho\neq 1. \tag{19}$$

Substituting (5) into (20), we have

$$I_R(\rho) = \frac{1}{(1-\rho)} log\Bigg[\int_0^\infty \Bigg((2\eta)^\rho(\lambda\theta + \beta\gamma x^{\beta-1}(1+\theta x))^\rho(1+\theta x)^{-\rho(\lambda+1)}e^{-\rho\gamma x^\beta}$$
$$\times \frac{[1-(1+\theta x)^{-\lambda}e^{-\gamma x^\beta}]^{\rho(\eta-1)}}{[1+[1-(1+\theta x)^{-\lambda}e^{-\gamma x^\beta}]^\eta]^{2\rho}}\Bigg)dx\Bigg].$$

With some algebraic manipulations, the R$\acute{e}$nyi entropy of the TIIHLLW distribution can be expressed as

$$I_R(\rho) = \frac{1}{(1-\rho)}\Bigg[\rho log(2\eta\theta\lambda) + log\Bigg(\sum_{l=0}^{\rho}\sum_{i,j,k=0}^{\infty}\binom{\rho}{l}\binom{1+i}{i}\binom{\eta(\rho+i)-\rho}{j}$$
$$\times \frac{(-1)^{i+j+k}[(\rho+j)\gamma]^k}{k!}\left(\frac{\gamma\beta}{\theta\lambda}\right)^l \frac{B(\beta(\rho+\lambda(\rho+j)-\beta(l+k)-1,\beta(l+k)+1-l)}{\theta^{\beta(l+k)+1-l}}\Bigg)\Bigg]. \tag{20}$$

Let $H_S[g(x)] = E[-log(g(X))]$ be the Shannon entropy of any continuous random variable defined as

$$H_S[g(x)] = E[-log(g(X))] = -\int_0^\infty g(x)log(g(x))dx. \tag{21}$$

By substituting (5) into (22), the Shannon entropy of the TIIHLLW distribution is obtained as follows.

$$H_S[g(x)] = E\Bigg[-log\Bigg(2\eta(\lambda\theta+\beta\gamma x^{\beta-1}(1+\theta x))(1+\theta x)^{-\lambda-1}e^{-\gamma x^\beta}\frac{[1-(1+\theta x)^{-\lambda}e^{-\gamma x^\beta}]^{\eta-1}}{[1+[1-(1+\theta x)^{-\lambda}e^{-\gamma x^\beta}]^\eta]^2}\Bigg)\Bigg].$$

After further algebraic evaluations, the Shannon entropy of the TIIHLLW distribution can be expressed as

$$H_S[g(x)] = -log(2\eta\theta\lambda) - \sum_{i=1}^\infty\sum_{j=0}^\infty\binom{i}{j}(-1)^i\left(\frac{\gamma\beta}{\theta\lambda}\right)^i\theta^j E\Big(X^{i(\beta-1)+j}\Big)$$
$$-(\beta+1)\sum_{i=1}^\infty\frac{(-1)^i\theta^i}{i}E\big(X^i\big) + \gamma E\big(X^\beta\big) + (\eta-1)\sum_{i=1}^\infty\sum_{j,l=0}^\infty\binom{\lambda i+j-1}{j}$$
$$\times\frac{(-1)^{j+l+1}(i\gamma)^l}{i!l!}\theta^j E\big(X^{l\beta+j}\big) - 2\sum_{i=1}^\infty\sum_{j,k,i=0}^\infty\binom{\lambda k+j-1}{j}\binom{\eta i}{k}$$
$$\times\frac{(-1)^{i+j+k+l}(\gamma k)^l}{i!l!}\theta^j E\big(X^{l\beta+j}\big). \tag{22}$$

Where $E(X^m)$ is defined in (12) and $m$ is a dummy quantity.

## 3. Parameter estimation and Simulation experiment

### 3.1. Parameter estimation

This section considers maximum likelihood estimation (MLE) as the method of parameter estimation for TIIHLLW distribution. Let $X\sim$TIIHLLW distribution having $F = (\theta,\lambda,\gamma,\beta,\eta)$ as parameter vector. For a random sample of complete observations given as $x_1, x_2, ..., x_n$ , the likelihood function of the random sample is given as;

$$\mathcal{L}_i(F) = \prod_{i=1}^{n} g(x_i, \theta, \lambda, \gamma, \beta, \eta) = \prod_{i=1}^{n}\left(2\eta(\lambda\theta + \beta\gamma x_i^{\beta-1}(1+\theta x_i))\frac{(1+\theta x_i)^{-\lambda-1}e^{-\gamma x_i^{\beta}}[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta-1}}{[1+[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta}]^2}\right). \quad (23)$$

The corresponding total log-likelihood function of (24) is

$$\begin{aligned}\mathcal{L}_n = \sum_{i=1}^{n}\log \mathcal{L}_i(F) &= n\log(2\eta) + \sum_{i=1}^{n}\log(\lambda\theta + \beta\gamma x_i^{\beta-1}(1+\theta x_i)) - (\lambda+1)\sum_{i=1}^{n}\log(1+\theta x_i) \\ &- \gamma\sum_{i=1}^{n} x_i^{\beta} + (\eta-1)\sum_{i=1}^{n}\log[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}] - 2\sum_{i=1}^{n}\log[1+[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta}].\end{aligned}$$

The first partial derivatives of $\mathcal{L}_n$ to $\theta, \lambda, \gamma, \beta, \eta$ respectively are obtained and equated to zero. The system of non-linear equations obtained is expressed as

$$\begin{aligned}\frac{\partial \mathcal{L}_n}{\partial \theta} &= -(\lambda+1)\sum_{i=1}^{n}\frac{x_i}{1+\theta x_i} + \sum_{i=1}^{n}\frac{\lambda + \beta\gamma x_i^{\beta}}{\lambda\theta + \beta\gamma x_i^{\beta-1}(1+\theta x_i)} \\ &+ (\eta-1)\sum_{i=1}^{n}\frac{\lambda x_i(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}}{1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}} - 2\eta\lambda\sum_{i=1}^{n}\frac{x_i(1+\theta x_i)^{-\lambda}[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]}{1+[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta}} = 0, \\ \frac{\partial \mathcal{L}_n}{\partial \lambda} &= \sum_{i=1}^{n}\frac{\theta}{\lambda\theta + \beta\gamma x_i^{\beta-1}(1+\theta x_i)} + (\eta-1)\sum_{i=1}^{n}\frac{(1+\theta x_i)^{-\lambda-1}e^{-\gamma x_i^{\beta}}\log(1+\theta x_i)}{1-(1+\theta x_i)^{-\lambda-1}e^{-\gamma x_i^{\beta}}} \\ &- \sum_{i=1}^{n}\log(1+\theta x_i) - 2\eta\sum_{i=1}^{n}\frac{(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta}\log(1+\theta x_i)}{1+[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta}} = 0, \\ \frac{\partial \mathcal{L}_n}{\partial \gamma} &= -\sum_{i=1}^{n} x_i^{\beta} + \beta\sum_{i=1}^{n}\frac{x_i^{\beta-1}(1+\theta x_i)}{\lambda\theta + \beta\gamma x_i^{\beta-1}(1+\theta x_i)} + (\eta-1)\sum_{i=1}^{n}\frac{x_i^{\beta}(1+\theta x_i)^{-\lambda-1}e^{-\gamma x_i^{\beta}}}{1-(1+\theta x_i)^{-\lambda-1}e^{-\gamma x_i^{\beta}}} \\ &- 2\eta\sum_{i=1}^{n}\frac{x_i^{\beta}(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta-1}}{1+[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta}} = 0, \\ \frac{\partial \mathcal{L}_n}{\partial \beta} &= -\gamma\sum_{i=1}^{n} x_i^{\beta}\log x_i - 2\eta\gamma\sum_{i=1}^{n}\frac{x_i^{\beta}(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta-1}\log x_i}{1+[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta}} \\ &+ \gamma\sum_{i=1}^{n}\frac{x_i^{\beta-1}(1+\theta x_i)(1+\beta\log x_i)}{\lambda\theta + \beta\gamma x_i^{\beta-1}(1+\theta x_i)} + \gamma(\eta-1)\sum_{i=1}^{n}\frac{x_i^{\beta}(1+\theta x_i)^{-\lambda-1}e^{-\gamma x_i^{\beta}}\log x_i}{1-(1+\theta x_i)^{-\lambda-1}e^{-\gamma x_i^{\beta}}} = 0, \\ \frac{\partial \mathcal{L}_n}{\partial \eta} &= \frac{n}{\eta} - 2\sum_{i=1}^{n}\frac{[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta}\log[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]}{1+[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}]^{\eta}} \\ &+ \sum_{i=1}^{n}\log[1-(1+\theta x_i)^{-\lambda}e^{-\gamma x_i^{\beta}}] = 0.\end{aligned}$$

The roots $(\theta, \lambda, \gamma, \beta, \eta)$ of the system of non-linear equations are obtained through numerical methods such as the Newton-Raphson iteration method. The roots are the parameter estimates of $\theta, \lambda, \gamma, \beta, \eta$.

Statistical inference for $\digamma = (\theta, \lambda, \gamma, \beta, \eta)$ in TIIHLLW distribution is carried out by deriving the asymptotic confidence intervals using Fisher's information matrix. These statistics obtained from the matrix are needed for the approximated confidence intervals for $\digamma$. The total observed information matrix $I_n(\digamma)$ is presented as

$$I_n(\digamma) = -\begin{pmatrix} \mathcal{L}_{n(\theta\theta)} & \mathcal{L}_{n(\theta\lambda)} & \mathcal{L}_{n(\theta\gamma)} & \mathcal{L}_{n(\theta\beta)} & \mathcal{L}_{n(\theta\eta)} \\ - & \mathcal{L}_{n(\lambda\lambda)} & \mathcal{L}_{n(\lambda\gamma)} & \mathcal{L}_{n(\lambda\beta)} & \mathcal{L}_{n(\lambda\eta)} \\ - & - & \mathcal{L}_{n(\gamma\gamma)} & \mathcal{L}_{n(\gamma\beta)} & \mathcal{L}_{n(\gamma\eta)} \\ - & - & - & \mathcal{L}_{n(\beta\beta)} & \mathcal{L}_{n(\beta\eta)} \\ - & - & - & - & \mathcal{L}_{n(\eta\eta)} \end{pmatrix}$$

Hence, $\hat{\theta} \pm Z_{\frac{\Psi}{2}}\sqrt{\hat{\mathcal{L}}_{n(\theta\theta)}}$, $\hat{\lambda} \pm Z_{\frac{\Psi}{2}}\sqrt{\hat{\mathcal{L}}_{n(\lambda\lambda)}}$, $\hat{\gamma} \pm Z_{\frac{\Psi}{2}}\sqrt{\hat{\mathcal{L}}_{n(\gamma\gamma)}}$, $\hat{\beta} \pm Z_{\frac{\Psi}{2}}\sqrt{\hat{\mathcal{L}}_{n(\beta\beta)}}$ and $\hat{\eta} \pm Z_{\frac{\Psi}{2}}\sqrt{\hat{\mathcal{L}}_{n(\eta\eta)}}$ give $100(1-\Psi)\%$ confidence interval for $\digamma = (\theta, \lambda, \gamma, \beta, \eta)$, where $Z_{\frac{\Psi}{2}}$ is the standard normal upper percentile.

### 3.2. Simulation experiment

Monte Carlo simulation experiment is carried out for parameters of TIIHLLW distribution to determine the performance of maximum likelihood estimators of unknown parameters of TIIHLLW distribution, $\digamma = (\theta, \lambda, \gamma, \beta, \eta)$, using some sets of values at different sample sizes. To achieve this objective, $N = 1000$ samples of sizes and $n = 25, 50, 100, 200, 400, 800$ are generated from the TIIHLLW distribution. By choosing a large $N$ for simulation, the true sampling distribution of data randomly generated from (10) is achieved based on the vector of parameter estimates, $\hat{\digamma}$. The sets of true parameter values considered for the simulation study are; Set I: $\theta = 0.2, \lambda = 1.0, \gamma = 0.5, \beta = 0.5, \eta = 0.5$, Set II: $\theta = 1.2, \lambda = 0.5, \gamma = 8.0, \beta = 0.3, \eta = 1.0$ and Set III: $\theta = 0.8, \lambda = 0.5, \gamma = 1.0, \beta = 0.5, \eta = 0.5$. The choice of N=1000 is based on the need to have a reasonably large N to yield a true sampling distribution of randomly generated data sets on which the MLE estimates of the distribution are based. Three quantities are computed in the simulation experiment. These quantities are

$$Mean_{\digamma} = \frac{1}{N}\sum_{i=1}^{N}\hat{\digamma}, \quad Bias_{\digamma} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\digamma} - \digamma) \quad \text{and} \quad RMSE_{\digamma} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\digamma} - \digamma)^2}.$$

The results of the simulation experiment are presented in Table 3.

The mean estimates in Table 3 become closer to the true parameters in Sets I, II, and III as the sample size $n$ increases; whereas, the bias and RMSEs deteriorate towards zero as $n$ becomes large. These properties suggest the efficiency and consistency of the maximum likelihood estimates of the TIIHLLW distribution.

## 4. Data Fittings

In this section, the application of the TIIHLLW distribution and other competing lifetime distributions to two real data sets are considered. The aim is to determine the flexibility and performance of the TIIHLLW distribution in comparison to other related existing distributions. Some discrepancy criteria for model comparison between the TIIHLLW distribution and other competing distributions are presented. The competing distributions are; the extended odd Lomax-Lomax (EOLxLx) distribution due to [2], Gumbel exponentiated Weibull logistic (GuEWL) distribution studied in [5], exponentiated additive Weibull (EAW) distribution proposed in [1], Kumaraswamy generalized power Lomax (KGPLx) distribution reported in [19], Nadarajah-Haghighi Lomax (NHLx) distribution developed by [20], Type II half logistic Lomax (TIIHLLx) distribution introduced by [12] and Weibull distribution proposed in [31].

**Table 3**. Simulation Results

| $F$ | n | Set I | | | Set II | | | Set III | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Mean | Bias | RMSE | Mean | Bias | RMSE | Mean | Bias | RMSE |
| $\theta$ | 25 | 0.632536 | 0.432536 | 2.764352 | 3.845336 | 2.645336 | 18.40852 | 0.716696 | -0.083304 | 3.878790 |
| | 50 | 0.555530 | 0.355530 | 2.502413 | 2.176381 | 0.976381 | 5.010832 | 0.956662 | 0.156662 | 3.722524 |
| | 100 | 0.336599 | 0.136599 | 0.478336 | 1.642784 | 0.442784 | 2.094876 | 1.085021 | 0.285021 | 2.647772 |
| | 200 | 0.240041 | 0.040041 | 0.090373 | 1.358828 | 0.158828 | 0.749863 | 0.798425 | -0.001575 | 0.206839 |
| | 400 | 0.210913 | 0.010913 | 0.014260 | 1.206031 | 0.006031 | 0.014550 | 0.821073 | 0.021073 | 0.226452 |
| | 800 | 0.202538 | 0.002538 | 0.001437 | 1.202046 | 0.002046 | 0.003348 | 0.803225 | 0.003225 | 0.002613 |
| $\lambda$ | 25 | 3.347442 | 2.347442 | 23.05667 | 4.961620 | 4.461620 | 36.65540 | 5.934159 | 5.434159 | 48.35291 |
| | 50 | 2.592627 | 1.592627 | 12.21634 | 2.735723 | 2.235723 | 17.63912 | 3.798946 | 3.298946 | 24.19149 |
| | 100 | 1.612263 | 0.612263 | 3.981125 | 1.230094 | 0.730094 | 4.691359 | 2.190951 | 1.690951 | 9.133722 |
| | 200 | 1.129621 | 0.129621 | 0.590977 | 0.816801 | 0.318601 | 2.090336 | 1.688443 | 1.188443 | 3.784379 |
| | 400 | 1.078632 | 0.078632 | 0.463838 | 0.513140 | 0.013140 | 0.069066 | 0.514805 | 0.014805 | 0.034695 |
| | 800 | 1.003918 | 0.003918 | 0.008014 | 0.504598 | 0.004598 | 0.016910 | 0.500399 | 0.000399 | 0.000781 |
| $\gamma$ | 25 | 1.304015 | 0.804015 | 1.837500 | 7.480209 | -0.519791 | 4.332088 | 2.891273 | 1.891273 | 9.040005 |
| | 50 | 1.096834 | 0.596834 | 1.433309 | 7.521989 | -0.478011 | 1.881334 | 1.695452 | 0.695452 | 1.478048 |
| | 100 | 0.710169 | 0.210169 | 0.458374 | 7.842561 | -0.157439 | 0.851189 | 1.320544 | 0.320544 | 0.574523 |
| | 200 | 0.611636 | 0.111636 | 0.192834 | 7.892637 | -0.107363 | 0.347586 | 1.115591 | 0.115591 | 0.124727 |
| | 400 | 0.549537 | 0.049537 | 0.060222 | 7.991959 | -0.008041 | 0.025863 | 1.108712 | 0.108712 | 0.026267 |
| | 800 | 0.509017 | 0.009017 | 0.006332 | 7.997536 | -0.002464 | 0.004858 | 1.001133 | 0.001133 | 0.000409 |
| $\beta$ | 25 | 0.667594 | 0.167594 | 0.502643 | 0.233373 | -0.066627 | 0.018771 | 0.806146 | 0.306146 | 4.105728 |
| | 50 | 0.699380 | 0.199380 | 0.882466 | 0.266331 | -0.033669 | 0.008644 | 0.443844 | -0.056156 | 0.183004 |
| | 100 | 0.541721 | 0.041721 | 0.091174 | 0.280536 | -0.019464 | 0.003930 | 0.448869 | -0.051131 | 0.050328 |
| | 200 | 0.516782 | 0.016782 | 0.036946 | 0.290645 | -0.009355 | 0.001415 | 0.475055 | -0.024945 | 0.016528 |
| | 400 | 0.494370 | -0.005630 | 0.009594 | 0.299565 | -0.000435 | 0.000007 | 0.494923 | -0.005077 | 0.003692 |
| | 800 | 0.497087 | -0.002923 | 0.001239 | 0.299863 | -0.000137 | 0.000001 | 0.499530 | -0.000157 | 0.000157 |
| $\eta$ | 25 | 3.358112 | 2.858112 | 25.72085 | 3.818015 | 2.818015 | 21.62441 | 4.046566 | 3.546566 | 29.59907 |
| | 50 | 3.045318 | 2.545318 | 22.91093 | 2.180656 | 1.180656 | 8.313844 | 3.692526 | 3.192526 | 28.65602 |
| | 100 | 1.260434 | 0.760434 | 5.831216 | 1.623261 | 0.623261 | 4.381452 | 1.573817 | 1.073817 | 8.162104 |
| | 200 | 0.821798 | 0.321798 | 2.244536 | 1.171341 | 0.171341 | 0.684518 | 0.744724 | 0.244724 | 0.948928 |
| | 400 | 0.595234 | 0.095234 | 0.295155 | 1.007930 | 0.007930 | 0.025152 | 0.545643 | 0.045643 | 0.119644 |
| | 800 | 0.511887 | 0.011887 | 0.012366 | 1.001436 | 0.001436 | 0.001649 | 0.501497 | 0.001497 | 0.001057 |

The measures of the discrepancy criteria for comparison of the TIIHLLW distribution with some competing distributions are presented. These include the maximized $\mathcal{L}_n$ , Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Cramer-von Mises ($W^\star$), Anderson-Darling ($A^\star$), Kolmogorov-Smirnov (K-S) statistics and the corresponding p-values. For comparison, the distribution with the smallest values for these measures is considered as the model that best fits the data set. The fitted densities for each data set are presented for further validation of the fitness of the distributions. The plots and computations are achieved with $R$ software.

### 4.1. COVID-19 infection in Nigeria

The first data set consists of the daily number of COVID-19 infected persons for 73 days in Nigeria between 20th October and 31st December 2020. The COVID-19 data set was collected from the National Center for Disease Control (NCDC) at http://covid19.ncdc.gov.ng/. The data set is as follows: 72, 37, 138, 77, 48, 62, 119, 113, 147, 150, 170, 162, 111, 72, 137, 155, 180, 223, 59, 300, 94, 152, 180, 212, 156, 112, 152, 157, 152, 236, 146, 143, 246, 155, 56, 168, 198, 169, 246, 110, 82, 145, 281, 122, 343, 324, 310, 318, 390, 550, 474, 675, 796, 617, 418, 201, 758, 930, 1145, 806, 920, 501, 356, 999, 1133, 1041, 784, 829, 838, 397, 749, 1016, 1031.

**Table 4**. Parameter estimates with standard errors for Competing Distributions

| Distributions | $\theta$ (std. error) | $\lambda$ (std. error) | $\gamma$ (std. error) | $\beta$ (std. error) | $\eta$ (std. error) |
|---|---|---|---|---|---|
| TIIHLLW | 0.183537 | 0.647192 | 0.004110 | 0.875212 | 19.607352 |
| | (0.625297) | (0.219976) | (0.003825) | (0.118916) | 49.142929 |
| EOLxLx | 17.794894 | 0.763624 | 0.301473 | 29.5871981 | 25.4175882 |
| | (54.881381) | 0.265424) | ( 0.944722) | (73.831853) | (79.738006) |
| GuEWL | 1.112289 | 0.267493 | 4.577597 | 0.473111 | 0.860813 |
| | (5.668049) | (0.126897) | (8.155536) | (2.716208) | (1.006339) |
| KGPLx | 0.042264 | 0.262201 | 1.843194 | 12.993170 | 6.826667 |
| | (0.392958) | (0.482043) | (2.007319) | (64.066661) | (23.104796) |
| EAW | 0.016432 | 0.083762 | 0.116564 | 0.538217 | 6.986570 |
| | (0.298527) | (0.358745) | (0.105494) | (0.119943) | (6.957892) |
| NHLx | 11.210763 | 2.687546 | 0.002558 | - | 0.217907 |
| | (7.032997) | (0.824773) | (0.000678) | | (0.079582) |
| TIIHLLx | 0.003932 | 2.103252 | - | - | 3.601468 |
| | (0.000780) | (0.321805) | | | (0.751823) |
| Weibull | - | - | 0.003312 | 0.973856 | - |
| | | | (0.000558) | (0.038727) | |

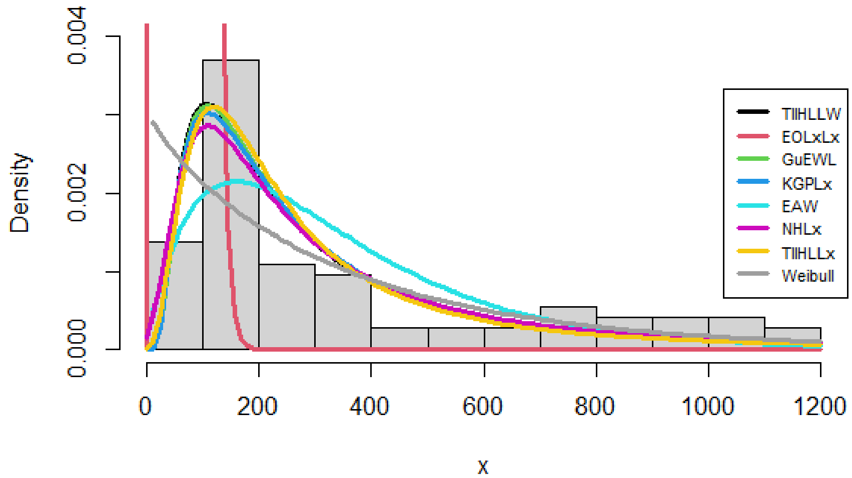
Table 4 presents the parameter estimates for TIIHLLW, EOLxLx, GuEWL, EAW, KGPLx, NHLx, TIIHLLx, and Weibull distributions. The standard error for each parameter estimate is in parentheses for each distribution.

**Table 5**. Discrepancy criteria for competing distributions in Data set 1

| Distributions | $-2LL$ | AIC | CAIC | HQIC | $W^*$ | $A^*$ | K-S (p-value) |
|---|---|---|---|---|---|---|---|
| TIIHLLW | 988.4939 | 998.494 | 999.389 | 1003.058 | 0.148879 | 0.933603 | 0.10063 (0.4505) |
| EOLxLx | 989.5397 | 999.539 | 1000.435 | 1004.104 | 0.199113 | 1.175065 | 0.12808 (0.1822) |
| GuEWL | 990.5930 | 1000.593 | 1001.488 | 1005.157 | 0.149415 | 0.957975 | 0.10207 (0.4324) |
| KGPLx | 990.835 | 1000.835 | 1001.731 | 1005.086 | 0.158601 | 1.001328 | 0.10842 (0.3574) |
| EAW | 993.001 | 1003.001 | 1003.905 | 1007.573 | 0.244504 | 1.438167 | 0.15782 (0.0527) |
| NHLx | 990.989 | 998.989 | 999.577 | 1002.640 | 0.195121 | 1.178589 | 0.10587 (0.3864) |
| TIIHLLx | 992.585 | 998.585 | 998.933 | 1001.324 | 0.159356 | 1.027604 | 0.10252 (0.4268) |
| Weibull | 1005.172 | 1009.172 | 1109.344 | 1010.988 | 0.395084 | 2.249722 | 0.13844 (0.1218) |

In Table 5, the discrepancy quantities for comparison amongst the competing distributions are presented. It is observed that the TIIHLLW distribution has the lowest $-2LL$, AIC, CAIC, HQIC, $W^*$, $A^*$ and K-S values in most of the discrepancy criteria. Also, the large p-value of the K-S criterion of the TIIHLLW distribution indicates that it provided a better fit than the other competing distributions. In Figure 2, the plots of the fitted densities for the competing models show that the proposed distribution provided a better fit for the data set.

**Figure 2**. Fitted densities for TIIHLLW distribution and other competing distributions




### 4.2. Percentage of the body fat ($\%Bfat$) for the 202 athletes

The second data set is about Australian Athletes reported in [8] which consists of 13 variables on 102 male and 100 female athletes collected at the Australian Institute of Sport. [13] used the heights for the 100 female athletes and the hemoglobin concentration levels for the 202 athletes to illustrate the application of a generalized skew two-piece skew-normal distribution. Also, [7] used the percentage of the hemoglobin blood cells for the male athletes to illustrate the applicability of an extended skew generalized normal distribution. We utilized the percentage of body fat for the 202 athletes to illustrate the effectiveness of the proposed TIILLW distribution over some competing distributions. The data set is given by: 19.75, 21.30, 19.88, 23.66, 17.64, 15.58, 19.99, 22.43, 17.95, 15.07, 28.83, 18.08, 23.30, 17.71, 18.77, 19.83, 25.16, 18.04,21.79, 22.25, 16.25, 16.38, 19.35, 19.20, 17.89, 12.20, 23.70, 24.69, 16.58, 21.47, 20.12, 17.51, 23.70, 22.39, 20.43, 11.29, 25.26, 19.39, 19.63, 23.11, 16.86, 21.32, 26.57, 17.93, 24.97, 22.62, 15.01, 18.14, 26.78, 17.22, 26.50, 23.01, 30.10, 13.93, 26.65, 35.52, 15.59, 19.61, 14.52, 11.47, 17.71, 18.48, 11.22, 13.61, 2.78, 11.85, 13.35, 11.77, 11.07, 21.30, 20.10, 24.88, 19.26, 19.51, 23.01, 8.07, 11.05, 12.39, 15.95, 9.91, 16.20, 9.02, 14.26, 10.48, 11.64, 12.16, 10.53, 10.15, 10.74, 20.86, 19.64, 17.07, 15.31, 11.07, 12.92, 8.45, 10.16, 12.55, 9.10, 13.46, 8.47, 7.68, 6.16, 8.56, 6.86, 9.40, 9.17, 8.54, 9.20, 11.72, 8.44, 7.19, 6.46, 9.00, 12.61, 9.03 ,6.96, 10.05, 9.56, 9.36, 10.81, 8.61, 9.53, 7.42, 9.79, 8.97, 7.49, 11.95, 7.35, 7.16, 8.77, 9.56, 14.53, 8.51, 10.64, 7.06, 8.87, 7.88, 9.20, 7.19, 6.06, 5.63, 6.59, 9.50, 13.97, 1.66, 6.43, 6.99, 6.00, 6.56, 6.03, 6.33, 6.82, 6.20, 5.93, 5.80, 6.56, 6.76, 7.22, 8.51, 7.72, 19.94, 13.91, 6.10, 7.52, 9.56, 6.06,7.35, 6.00, 6.92, 6.33, 5.90, 8.84, 8.94,6.53, 9.40, 8.18, 17.41, 18.08, 9.86,7.29, 18.72, 10.12, 19.17, 17.24, 9.89, 13.06, 8.84, 8.87, 14.69, 8.64, 14.98, 7.82, 8.97, 11.63, 13.49, 10.25, 11.79, 10.05, 8.51, 11.50, 6.26.

Table 6 presents the parameter estimates for TIIHLLW, EOLxLx, GuEWL, EAW, KGPLx, NHLx, TIIHLLx, and Weibull distributions. The standard error for each parameter estimate is in parentheses for each distribution. In Table 7, it is observed that the TIIHLLW distribution has the lowest $-2LL$, AIC, CAIC, HQIC, $W^*$, $A^*$ and K-S values in the discrepancy criteria. Also, the large p-value of the K-S criterion indicates that the TIIHLLW distribution provides a better fit than the other competing distributions. In Figure 3, the plots of the fitted densities for the competing models show that the proposed distribution is flexible to analyse the data sets.

**Table 6**. Parameter estimates with standard errors for competing distributions

| Distributions | $\theta$ (std. error) | $\lambda$ (std. error) | $\gamma$ (std. error) | $\beta$ (std. error) | $\eta$ (std. error) |
|---|---|---|---|---|---|
| TIIHLLW | 0.421271<br>(0.577471) | 1.167837<br>(0.356865) | 0.005517<br>(0.003182) | 1.832981<br>(0.158412) | 14.855778<br>(15.373199) |
| EOLxLx | 0.203463<br>(0.161771) | 1.872590<br>(0.580070) | 15.091316<br>(25.227711) | 12.132548<br>(7.729646) | 24.672130<br>(42.008424) |
| GuEWL | 1.034281<br>(0.017754) | 0.994796<br>(0.018529) | 8.169108<br>(0.487159) | 6.963616<br>(0.628734) | 4.615283<br>(0.350813) |
| KGPLx | 0.111418<br>(0.211973) | 2.642077<br>(8.430374) | 0.794612<br>(0.728466) | 9.479664<br>(13.542864) | 5.910384<br>(12.108740) |
| EAW | 0.794316<br>(2.268716) | 0.168395<br>(1.090761) | 0.084252<br>(0.440313) | 1.085332<br>(0.837028) | 8.122717<br>(18.475869) |
| NHLx | 3.439282<br>(1.107115) | 4.302869<br>(0.949331) | 0.004422<br>(0.001276) | - | 0.369353<br>(0.098416) |
| TIIHLLx | 0.018794<br>(0.006211) | 0.168395<br>(2.912489) | - | - | 8.122717<br>(1.400179) |
| Weibull | - | - | 0.003667<br>(0.000558) | 2.080376<br>(0.055678) | - |

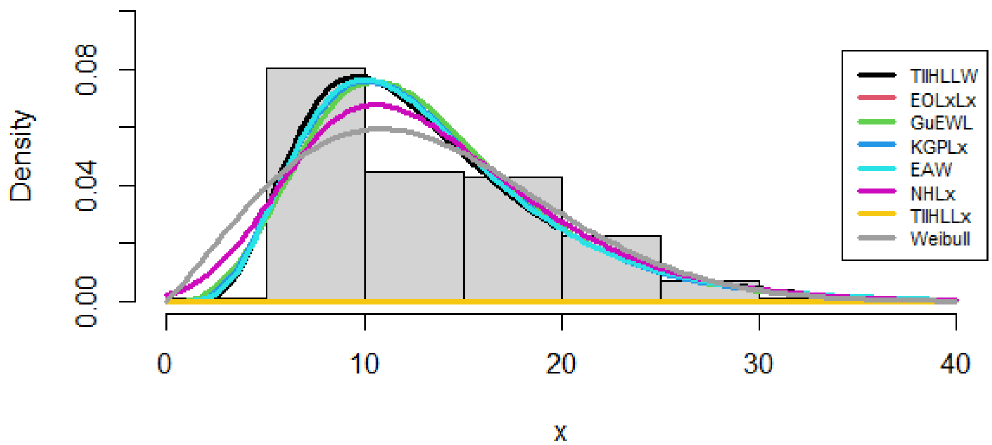
**Table 7**. Discrepancy criteria for competing distributions in Data set II

| Distributions | $-2LL$ | AIC | CAIC | HQIC | $W^*$ | $A^*$ | K-S (p-value) |
|---|---|---|---|---|---|---|---|
| TIIHLLW | 1273.210 | 1283.210 | 1283.516 | 1289.903 | 0.328264 | 2.051631 | 0.08399<br>(0.1157) |
| EOLxLx | 1273.773 | 1283.773 | 1284.080 | 1290.466 | 0.367079 | 2.225904 | 0.08983<br>(0.0768) |
| GuEWL | 1279.384 | 1289.384 | 1289.690 | 1296.077 | 0.465239 | 2.779119 | 0.09525<br>(0.0512) |
| KGPLx | 1277.384 | 1287.384 | 1287.690 | 1294.076 | 0.416036 | 2.524243 | 0.08690<br>(0.0946) |
| EAW | 1275.601 | 1288.601 | 1285.907 | 1292.293 | 0.516984 | 3.011438 | 0.16213<br>(4.887E-05) |
| NHLx | 1284.127 | 1292.127 | 1292.330 | 1297.481 | 0.546080 | 3.163976 | 0.08985<br>(0.0767) |
| TIIHLLx | 1278.894 | 1284.894 | 1285.015 | 1288.909 | 0.341289 | 2.227325 | 0.09294<br>(0.0610) |
| Weibull | 1292.984 | 1296.984 | 1297.045 | 1299.661 | 0.605086 | 3.468811 | 0.122550<br>(0.0045) |

## 5. Conclusion

The Type II half Logistic-Lomax-Weibull (TIIHLLW) distribution as a new five-parameter lifetime distribution has been proposed in this paper. The hazard function of the TIIHLLW distribution exhibits flexible properties in handling monotonic and non-monotonic lifetime data. Explicit expressions were obtained for the moments, quantile function, mean deviations, residual and reversed residual lifetime, order statistics and distribution as well as measures of entropies. Simulation experiments and parameter estimation were presented for the TIIHLLW distribution. The results of the simulation experiment for the TIIHLLW distribution are presented and discussed. The application of the TIIHLLW distribution to two real data sets validates the flexibility and

**Figure 3**. Fitted densities for TIIHLLW distribution and other competing distributions




superiority of the TIIHLLW distribution over other competing distributions in the study. It is hoped that the proposed distribution will attract wider applications in lifetime analysis.

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