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On geometric properties of generalized concave meromorphic functions

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Abstract: In this work, we introduce a generalized class of concave meromorphic functions denoted as $\mathcal{K}_0^n(\zeta)$ defined by Salagean differential operator $\mathcal{D}^n$, which is an operator defined on the concave meromorphic function $g(z)$, $\mathcal{D}^n g(z) = \mathcal{D}(\mathcal{D}^{n-1}g(z)), \{n \in \mathbb{N} \cup \{0\}\}$, and study some of the properties namely; inclusion, integral representation, closure under an integral operator, sufficient condition, coefficient inequality, growth and distortion of this class.

Key words: Salagean operator, concave and meromorphic functions

1. Introduction

Let g be define as

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k. \tag{1}$$

meromorphic function with a simple pole at the origin, the unit disk be denoted by $\{\mathbb{U} = |z| < 1\}$ and concave domain which is the exterior of a closed convex domain be denoted by $\mathbb{E}$. We say the function g of the form (1), is concave if it maps the $\mathbb{U}$ to $\mathbb{E}$, denoted by $\mathcal{K}_0$ satisfying the following inequality as define below:

Definition 1.1. [16] The function $g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$ belong to the class $\mathcal{K}_o$ if it satisfies the inequality

$$Re\left(1 + z\frac{g^{''}(z)}{g^{'}(z)}\right) < 0, z \in \mathbb{U}. \tag{2}$$

For more details of concave univalent functions and the types, (see{[1],[2],[3],[4],[6],[7] [19]}).
The integral representation of the functions in the class $\mathcal{K}_0$ was first considered in [15, 16] as stated below:

Theorem 1.1. *[15] The function $g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$ belong to the class function $\mathcal{K}_o$ if and only if there exists a positive measure $\mu(t)$ and $\int_{-\pi}^{\pi} d\mu(t) = 1$ and $\int_{-\pi}^{\pi} e^{-it} d\mu(t) = 0$, such that for $z \in \mathbb{U}$.*

$$g^{'}(z) = -\frac{1}{z} exp \int_{-\pi}^{\pi} 2log(1 - e^{it}z) d\mu(t). \tag{3}$$

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Theorem 1.2. *[16] The function* $g = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$ *be belong to the class* $\mathcal{K}_o$ *if there exists a function*

$$\varphi : \mathbb{U} \to \mathbb{U}, with \varphi(0) = 0 \tag{4}$$

holomorphic in $\mathbb{U}$ *, such that for* $z \in \mathbb{U}$

$$g^{'}(z) = -\frac{1}{z} exp \int_{-\pi}^{\pi} 2log(1 - e^{it}z) d\mu(t). \tag{5}$$

Conversely, for any holomorphic function $\varphi : \mathbb{U} \to \mathbb{U}$*, with* $\varphi(0) = 0$*, there exists a function* $g \in \mathcal{K}_o$*.*

In [13, 14, 16], the inequality (2) was shown to be the necessary and sufficient condition of the concave meromorphic function of the form (1) and also deduced the coefficient inequality

$$|a_1|^2 + 3|a_2| \leq 1 \tag{6}$$

by applying an invariant form of Schwarz's lemma involving with Schwarzian derivative and in [4], Al-Kaseasbeh, estimated a_k for $k = 2, 3, \cdots$ for the function $g(z)$ satisfying inequality (2).
Using the Salagean differential operator denoted as $\mathcal{D}^n$ is defined as $\mathcal{D}^0 g(z) = g(z)$, $\mathcal{D}^1 g(z) = z g^{'}(z)$, $\mathcal{D}^n g(z) = D(D^{n-1} g(z)), \{n \in \mathbb{N} \cup \{0\}\}$ and its integral operator define as $\mathcal{I}^0 g(z) = g(z)$, $\mathcal{I}^1 g(z) = \int_0^z \frac{g(t)}{t} dt$, $\mathcal{I}^n g(z) = \mathcal{I}(\mathcal{I}^{n-1} g(z)), \{n \in \mathbb{N} \cup \{0\}\}$, both appeared in [17].
The new generalized class of concave meromorphic functions is define as follows:

Definition 1.2. The function $g : \mathbb{U} \to \mathbb{E}$, $g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$ belong to the class of concave meromorphic univalent function of order ζ, denoted as $\mathcal{K}_0^n(\zeta)$, for $n \in \mathbb{N}$ and $0 \leq \zeta < 1$, if and only if it satisfies the inequality

$$Re\left(\frac{\mathcal{D}^{n+1} g(z)}{\mathcal{D}^n g(z)}\right) < -\zeta, z \in \mathbb{U}. \tag{7}$$

Similarly, the class $\mathcal{K}_0^n(\zeta)$ can be written as

$$-Re\left(\frac{\mathcal{D}^{n+1} g(z)}{\mathcal{D}^n g(z)}\right) > \zeta, z \in \mathbb{U}. \tag{8}$$

Our focus in this work is to study the concave meromorphic function using the Salagean derivative denoted by $\mathcal{K}_0^n(\zeta)$ and establish some of it geometric properties.

2. Preliminary Lemmas

Lemma 2.1. *[5] Let* $\mathcal{P}$ *be holomorphic in* $\mathbb{U}$ *with* $\mathcal{P}(0) = 1$ *and suppose that*

$$Re\left(\frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}\right) > \frac{3\zeta - 1}{2\zeta}, \in \mathbb{U}.$$

Then $Re\mathcal{P}(z) > 2^{1-1/\zeta}$*,* $1/2 \leq \zeta < 1$*,* $z \in \mathbb{U}$ *and the constant* $2^{1-1/\zeta}$ *is the best possible.*

Lemma 2.2. *[9, 11] Let* $\mathcal{P}(z)$ *be analytic in* $\mathbb{U}$, $\mathcal{P}(0)=1$ *and suppose that*

$$Re\left\{\mathcal{P}(z)-\frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}\right\}>\zeta,(z\in\mathbb{U},0\leq\zeta<1).$$

Then $Re\mathcal{P}(z)>\zeta$ *in* $\mathbb{U}$.

Lemma 2.3. *[10] Let* $\mathcal{P}(z)=1+c_1z+c_2z^2+\cdots$, *be analytic in* $\mathbb{U}$ *and* $\{\zeta:0\leq\zeta<1\}$ *be a positive real number. Suppose that* $\{r:0<r<1\}$, *such that*

$$\min_{|z|\leq r}Re\{\mathcal{P}(z)\}=\min_{|z|\leq r}|\mathcal{P}(z)|. \tag{9}$$

$$Re\frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}>\zeta-1,z\in\mathbb{U}, \tag{10}$$

for $0<\zeta\leq 1/2$,
and

$$Re\frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}>\zeta/2-1,z\in U. \tag{11}$$

for $1/2<\zeta<1$,
Then

$$Re\{\mathcal{P}(z)\}>\zeta,z\in U. \tag{12}$$

Lemma 2.4. *[18] Let* $\mathcal{P}(z)$ *be regular and satisfy* $Re\mathcal{P}(z)>\zeta$, $0\leq\zeta<1$ *in* $|z|<1$ *and let* $\mathcal{P}(0)=1$. *Then we have*

$$\mathcal{P}(z)=\frac{1+(2\zeta-1)z\phi(z)}{1+z\phi(z)} \tag{13}$$

where $\phi(z)$ *is any regular function in* $|z|<1$, *satisfying* $|\phi(z)|<1$ *in* $|z|<1$ *and any function* $\mathcal{P}(z)$ *given by the above formula is regular and satisfies* $Re\mathcal{P}(z)>\zeta$ *in* $|z|<1$.

Lemma 2.5. *[8, 15] Let* $w(z)$ *be non-constant regular in* $\{\ z:|z|<1\}$ $w(0)=0$. *If* $w(z)$ *attains its maximum value on the circle* $|z|=r<1$ *at* z_0, *we have* $z_0w^{'}(z_0)=kw(z_0)$, *where* k *is a real number* , $k\geq 1$.

3. Main Results

Theorem 3.1. *For* $n\in\mathbb{N}$ *and* $0\leq\zeta<1$. *Then* $\mathcal{K}_0^{n+1}(\zeta)\subset\mathcal{K}_0^{n}(\zeta)$.

Proof. The function $g(z)\in\mathcal{K}_0^n(\zeta)$, if $\mathcal{P}\in\mathcal{P}(\zeta)$ so that

$$-\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^ng(z)}=\mathcal{P}(z). \tag{14}$$

By differentiating (14), we obtain

$$\frac{(\mathcal{D}^ng(z))^{'}\mathcal{D}^{n+1}g(z)}{(\mathcal{D}^ng(z))^2}-\frac{(\mathcal{D}^{n+1}g(z))^{'}}{\mathcal{D}^ng(z)}=\mathcal{P}^{'}(z) \tag{15}$$

$$\frac{z(\mathcal{D}^n g(z))^{'}\mathcal{D}^{n+1}g(z)}{(\mathcal{D}^n g(z))^2} - \frac{z(\mathcal{D}^{n+1}g(z))^{'}}{\mathcal{D}^n g(z)} = z\mathcal{P}^{'}(z). \tag{16}$$

Divide through by $\mathcal{P}(z)$

$$\frac{z(\mathcal{D}^{n+1}g(z))^{'}}{\mathcal{D}^{n+1}g(z)} - \frac{z(\mathcal{D}^n g(z))^{'}}{\mathcal{D}^n g(z)} = \frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}. \tag{17}$$

By the relation $z(\mathcal{D}^{n+1}g(z))^{'} = \mathcal{D}^{n+2}g(z)$ and $z(\mathcal{D}^n g(z))^{'} = \mathcal{D}^{n+1}g(z)$, then (17) becomes

$$\frac{\mathcal{D}^{n+2}g(z)}{\mathcal{D}^{n+1}g(z)} - \frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)} = \frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)} \tag{18}$$

which implies that

$$\frac{\mathcal{D}^{n+2}g(z)}{\mathcal{D}^{n+1}g(z)} = \frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)} + \frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}. \tag{19}$$

From (19), we have that

$$\frac{\mathcal{D}^{n+2}g(z)}{\mathcal{D}^{n+1}g(z)} = -\mathcal{P}(z) + \frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)} \tag{20}$$

$$-\frac{\mathcal{D}^{n+2}g(z)}{\mathcal{D}^{n+1}g(z)} = \mathcal{P}(z) - \frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}. \tag{21}$$

Since

$$Re\left\{\mathcal{P}(z) - \frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}\right\} > \zeta$$

by Lemma 2.2. Then $-Re\left(\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)}\right) > \zeta$ and we have the inclusion. □

Theorem 3.2. *If* $g(z) \in \mathcal{K}_0^n(\zeta)$, *and satisfies*

$$Re\left(\frac{\mathcal{D}^{n+2}g(z)}{\mathcal{D}^{n+1}g(z)} - \frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)}\right) > \frac{\zeta - 1}{2\zeta}.$$

Then $-Re(\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)}) > 2^{1-1/\zeta},\ 1/2 \le \zeta < 1,\ z \in \mathbb{U}$.

Proof. The function $g(z) \in \mathcal{K}_0^n(\zeta)$, if $\mathcal{P} \in \mathcal{P}(\zeta)$, then

$$-\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)} = \mathcal{P}(z). \tag{22}$$

By differentiating (22), we obtain

$$\frac{(\mathcal{D}^n g(z))^{'}\mathcal{D}^{n+1}g(z)}{(\mathcal{D}^n g(z))^2} - \frac{(\mathcal{D}^{n+1}g(z))^{'}}{\mathcal{D}^n g(z)} = \mathcal{P}^{'}(z) \tag{23}$$

$$\frac{z(\mathcal{D}^n g(z))^{'}\mathcal{D}^{n+1}g(z)}{(\mathcal{D}^n g(z))^2} - \frac{z(\mathcal{D}^{n+1}g(z))^{'}}{\mathcal{D}^n g(z)} = z\mathcal{P}^{'}(z) \tag{24}$$

$$\frac{z(\mathcal{D}^{n+1}g(z))^{'}}{\mathcal{D}^{n+1}g(z)} - \frac{z(\mathcal{D}^{n}g(z))^{'}}{\mathcal{D}^{n}g(z)} = \frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}. \tag{25}$$

By the fact that $z(\mathcal{D}^{n+1}g(z))^{'} = \mathcal{D}^{n+2}g(z)$ and $z(\mathcal{D}^{n}g(z))^{'} = \mathcal{D}^{n+1}g(z)$, then (25) becomes

$$\frac{\mathcal{D}^{n+2}g(z)}{\mathcal{D}^{n+1}g(z)} - \frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^{n}g(z)} = \frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}. \tag{26}$$

By the condition of the theorem,

$$Re\left(1 + \frac{z\mathcal{P}^{'}(z)}{\mathcal{P}(z)}\right) = Re\left(\frac{\mathcal{D}^{n+2}g(z)}{\mathcal{D}^{n+1}g(z)} - \frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^{n}g(z)} + 1\right) > \frac{3\zeta - 1}{2\zeta} \tag{27}$$

which is equivalent to

$$Re\left(\frac{\mathcal{D}^{n+2}g(z)}{\mathcal{D}^{n+1}g(z)} - \frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^{n}g(z)}\right) > \frac{\zeta - 1}{2\zeta}. \tag{28}$$

Thus by Lemma 2.1, $Re\mathcal{P}(z) > 2^{1/\zeta - 1}$, $1/2 \leq \zeta < 1$, which concludes the result. □

Theorem 3.3. *Let* $f \in \mathcal{K}_0^n(\zeta)$, $n \in \mathbb{N}$ *and* $0 \leq \zeta < 1$. *Then*

$$g(z) = \mathcal{I}_n\left\{\frac{1}{z}\exp\left\{(2-2\zeta)\int_0^z \frac{\phi(t)}{1+t\phi(t)}dt\right\}\right\}. \tag{29}$$

where $\phi(z)$ *is regular in* $|z| < 1$ *with* $|\phi(z)| < 1$.

Proof. Let $g \in \mathcal{K}_0^n(\zeta)$, then by Lemma 2.4

$$\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^{n}g(z)} = -\frac{1 + (2\zeta - 1)z\phi(z)}{1 + z\phi(z)}. \tag{30}$$

From the relation $z(\mathcal{D}^{n}g(z))^{'} = \mathcal{D}^{n+1}g(z)$, we obtain

$$\frac{z(\mathcal{D}^{n}g(z))^{'}}{\mathcal{D}^{n}g(z)} = -\frac{1 + (2\zeta - 1)z\phi(z)}{1 + z\phi(z)}. \tag{31}$$

$$\frac{z(\mathcal{D}^{n}g(z))^{'}}{\mathcal{D}^{n}g(z)} + 1 = \frac{(2-2\zeta)z\phi(z)}{1 + z\phi(z)}. \tag{32}$$

$$\frac{z(\mathcal{D}^{n}g(z))^{'}}{\mathcal{D}^{n}g(z)} + \frac{1}{z} = \frac{(2-2\zeta)\phi(z)}{1 + z\phi(z)}. \tag{33}$$

We can have that

$$\frac{d}{dz}(\log z\mathcal{D}^{n}g(z)) = \frac{(2-2\zeta)\phi(z)}{1 + z\phi(z)}. \tag{34}$$

Which gives

$$\mathcal{D}^{n}g(z)) = \frac{1}{z}\exp\left\{(2-2\zeta)\int_0^z \frac{\phi(t)}{1+t\phi(t)}\right\}. \tag{35}$$

Equation (29) can easily be obtained from (35). □

Corollary 3.1. . *If* $n = 1$, *then*

$$g(z) = \int_0^z \left\{ \frac{1}{s^2} \left\{ \exp \left\{ (2 - 2\zeta) \int_0^s \frac{\phi(t)}{1 + t\phi(t)} dt \right\} \right\} \right\}. \tag{36}$$

Corollary 3.2. *[1]. If* $n = 1$, $\zeta = 0$, *then*

$$g(z) = \int_0^z \left\{ \frac{1}{s^2} \left\{ \exp \left\{ \int_0^s \frac{2\phi(t)}{1 + t\phi(t)} dt \right\} \right\} \right\}. \tag{37}$$

Theorem 3.4. *Let*

$$G(z) = \frac{c}{z^{c+1}} \int_0^z t^c g(t) dt. \tag{38}$$

If $G(z)$ *satisfies the condition*

$$Re \frac{\mathcal{D}^{n+1} g(z)}{\mathcal{D}^n g(z)} < \frac{1 - \zeta}{2c + 2 - 2\zeta} - \zeta \tag{39}$$

for $n \in \mathbb{N}$, $0 \le \zeta < 1$ *and* $c > 0$. *Then* $G(z) \in \mathcal{K}_0^n(\zeta)$.

Proof. Let

$$\frac{\mathcal{D}^{n+1} g(z)}{\mathcal{D}^n g(z)} = -\mathcal{P}(z). \tag{40}$$

From (38), we have

$$z^{c+1} G(z) = c \int_0^z t^c g(t) dt \tag{41}$$

$$(c + 1) z^c G(z) + z^{c+1} G^{'}(z) = c z^c g(z) \tag{42}$$

$$(c + 1) G(z) + z G^{'}(z) = c g(z) \tag{43}$$

$$(c + 1) \mathcal{D}^n G(z) + z (\mathcal{D}^n G(z))^{'} = c \mathcal{D}^n g(z) \tag{44}$$

$$(c + 1) \mathcal{D}^{n+1} G(z) + z (\mathcal{D}^{n+1} G(z))^{'} = c \mathcal{D}^{n+1} g(z) \tag{45}$$

$$c \mathcal{D}^{n+1} g(z) = -(c + 1) [\mathcal{P}(z) \mathcal{D}^n G(z)] - z [\mathcal{P}(z) \mathcal{D}^n G(z)]^{'} \tag{46}$$

$$c \mathcal{D}^{n+1} g(z) = -(c + 1) [\mathcal{P}(z) \mathcal{D}^n G(z)] - z \mathcal{P}^{'}(z) \mathcal{D}^n G(z) - \mathcal{P}(z) z [\mathcal{D}^n G(z)]^{'} \tag{47}$$

$$c \mathcal{D}^{n+1} g(z) = -(c + 1) [\mathcal{P}(z) \mathcal{D}^n G(z)] - z \mathcal{P}^{'}(z) \mathcal{D}^n G(z) + \mathcal{P}(z) \mathcal{D}^{n+1} G(z) \tag{48}$$

$$c \mathcal{D}^{n+1} g(z) = -[(c + 1) \mathcal{P}(z) \mathcal{D}^n G(z) - z \mathcal{P}^{'}(z) + \mathcal{P}^2(z)] \mathcal{D}^n G(z). \tag{49}$$

Also

$$c \mathcal{D}^n g(z) = (c + 1) \mathcal{D}^n G(z) + z (\mathcal{D}^n G(z))^{'} \tag{50}$$

$$c \mathcal{D}^n g(z) = (c + 1) \mathcal{D}^n G(z) + \mathcal{D}^{n+1} G(z) \tag{51}$$

$$c \mathcal{D}^n g(z) = [(c + 1) - \mathcal{P}(z)] D^n G(z) \tag{52}$$

$$\frac{c\mathcal{D}^{n+1}g(z)}{c\mathcal{D}^{n}g(z)}=\frac{-[(c+1)\mathcal{P}(z)-z\mathcal{P}^{'}(z)+\mathcal{P}^{2}(z)]\mathcal{D}^{n}G(z)}{[(c+1)-\mathcal{P}(z)]\mathcal{D}^{n}G(z)} \tag{53}$$

$$\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^{n}g(z)}=\frac{-[(c+1)-\mathcal{P}(z)]\mathcal{P}(z)-z\mathcal{P}^{'}(z)}{[(c+1)-\mathcal{P}(z)]} \tag{54}$$

$$\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^{n}g(z)}=-\mathcal{P}(z)-\frac{z\mathcal{P}^{'}(z)}{[(c+1)-\mathcal{P}(z)]}. \tag{55}$$

From Lemma (2.4), let

$$p(z)=\frac{1+(2\zeta-1)w(z)}{1+w(z)} \tag{56}$$

where $w(z)=z\phi(z)$, $w(0)=0$ and $|w(z)|<1$,

$$\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^{n}g(z)}=-\left\{\frac{1+(2\zeta-1)w(z)}{1+w(z)}+\frac{(2\zeta-2)zw^{'}(z)}{(1+w(z))[c+(2+c-2\zeta)]w(z)}\right\}. \tag{57}$$

By Lemma 2.5, there exits $k\geq 1$ such that $z_0w^{'}(z_0)=kw(z_0)$, we obtain

$$\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^{n}g(z)}=-\left\{\frac{1+(2\zeta-1)w(z_0)}{1+w(z_0)}+\frac{(2\zeta-2)kw(z_0)}{(1+w(z_0))[c+(2+c-2\zeta)]w(z_0)}\right\}. \tag{58}$$

So that

$$Re\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^{n}g(z)}\geq\frac{1-\zeta}{2c+2-2\zeta}-\zeta>0, \tag{59}$$

which is a contradiction for $|w(z)|<1$, then $G(z)\in\mathcal{K}_0^n(\zeta)$. □

Corollary 3.3. *If $n=1$, then*

$$Re\left(1+\frac{zg^{''}(z)}{g(z)}\right)<\frac{1-\zeta}{2c+2-2\zeta}-\zeta. \tag{60}$$

Corollary 3.4. *If $n=1$, $\zeta=0$, then*

$$Re\left(1+\frac{zg^{''}(z)}{g(z)}\right)<\frac{1}{2+2c}. \tag{61}$$

Corollary 3.5. *[12]. If $n=1$, $\zeta=0$ and $c=1$ then*

$$Re\left(1+\frac{zg^{''}(z)}{g(z)}\right)<\frac{1}{4}. \tag{62}$$

Theorem 3.5. *For $n\in\mathbb{N}$ and $0\leq\zeta<1$. The class $\mathcal{K}_0^n(\zeta)$, is a convex family of concave meromorphic univalent functions.*

Proof. Let $g(z)$ and $k(z)$ be in the class $\mathcal{K}_0^n(\zeta)$. For $t \in (0,1)$, it suffices to show that the function $h(z) = (1-t)g(z) + tk(z)$ is in the class $\mathcal{K}_0^n(\zeta)$.

$$-Re\frac{\mathcal{D}^{n+1}h(z)}{\mathcal{D}^n h(z)} = \frac{(1-t)[\frac{(-1)^{n+1}}{z} + \sum_{k=1}^{\infty} k^{n+1}a_k z^k] + t[\frac{(-1)^{n+1}}{z} + \sum_{k=1}^{\infty} k^{n+1}b_k z^k]}{(1-t)[\frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k] + t[\frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n b_k z^k]}. \tag{63}$$

$$-Re\frac{\mathcal{D}^{n+1}h(z)}{\mathcal{D}^n h(z)} = \frac{\frac{(-1)^{n+1}}{z} + \sum_{k=1}^{\infty} k^{n+1}[(1-t)a_k + tb_k z^k]z^k}{\frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n[(1-t)a_k + tb_k]z^k}. \tag{64}$$

Since $g(z), k(z) \in \mathcal{K}_0^n(\zeta)$. This implies that $h(z) = \frac{1}{z} + \sum_{k=1}^{\infty}[(1-t)a_k + tb_k]z^k \in \mathcal{K}_0^n(\zeta)$. Therefore

$$-Re\frac{\mathcal{D}^{n+1}h(z)}{\mathcal{D}^n h(z)} > \zeta. \tag{65}$$

□

Theorem 3.6. *Let* $n \in \mathbb{N}$, $0 \le \zeta < 1$. *Suppose that* $g(z)$ *satisfies the condition*

$$\min_{|z|\le r} Re\left\{-\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)}\right\} = \min_{|z|\le r}\left|-\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)}\right|, \tag{66}$$

for arbitrary $(0 < r < 1)$

$$Re\frac{\mathcal{D}^{n+2}g(z)}{\mathcal{D}^{n+1}g(z)} < Re\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)} - \zeta + 1, \tag{67}$$

for $0 < \zeta \le 1/2$.
and

$$Re\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^{n+1}f(z)} < Re\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)} - \frac{\zeta}{2} + 1, z \in U, \tag{68}$$

for $1/2 < \zeta < 1$.
Then $f(z) \in \mathcal{K}_0^n(\zeta)$.

Proof. Let

$$-\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)} = \mathcal{P}(z) \tag{69}$$

then

$$\frac{z\mathcal{P}'(z)}{\mathcal{P}(z)} = -\frac{\mathcal{D}^{n+2}f(z)}{\mathcal{D}^{n+1}g(z)} + \frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)} \tag{70}$$

By Lemma 2.3, condition (67) and (68), we have that

$$-Re\left(\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)}\right) > \zeta. \tag{71}$$

□

Theorem 3.7. *Let $n \in \mathbb{N}$ and $g(z)$ be of the form (1). For $0 \leq \zeta < 1$, then*

$$\sum_{k=1}^{\infty} k^n(\zeta + k)|a_k| \leq 1 - \zeta. \tag{72}$$

if and only $f \in \mathcal{K}^n(\zeta)$.

Proof. Suppose that the condition (72) holds for $(0 \leq \zeta < 1)$, it is sufficient to show that $|1-\zeta+\alpha| \leq |1+\beta-\alpha|$ where $Re(-\alpha) \geq \zeta$ which implies that

$$\left|1 - \zeta + \left(-\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)}\right)\right| \leq \left|1 + \zeta - \left(-\frac{\mathcal{D}^{n+1}g(z)}{\mathcal{D}^n g(z)}\right)\right|.$$

From the relation that $z(\mathcal{D}^n g(z))' = \mathcal{D}^{n+1}g(z)$, then

$$\left|1 - \zeta + \left(-\frac{z(\mathcal{D}^n g(z))'}{\mathcal{D}^n g(z)}\right)\right| \leq \left|1 + \zeta - \left(-\frac{z(\mathcal{D}^n g(z))'}{\mathcal{D}^n g(z)}\right)\right|$$

$$\left|1 - \zeta + \left(-\frac{z(\mathcal{D}^n g(z))'}{\mathcal{D}^n g(z)}\right)\right| - \left|1 + \zeta - \left(-\frac{z(\mathcal{D}^n g(z))'}{\mathcal{D}^n g(z)}\right)\right| \leq 0$$

$$\left|(1-\zeta)\mathcal{D}^n g(z) + (-z(\mathcal{D}^n g(z))')\right| - \left|(1+\zeta)\mathcal{D}^n g(z) - (-z(\mathcal{D}^n f(z))'\right| \leq 0$$

$$= \left|(1-\zeta)\left(\frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k\right) + \frac{(-1)^n}{z} - \sum_{k=1}^{\infty} k^{n+1} a_k z^k\right|$$

$$- \left|(1+\zeta)\left(\frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k\right) - \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^{n+1} a_k z^k\right|$$

$$\leq (2-\zeta)|-1|^n + \sum_{k=1}^{\infty}[(1-\zeta)k^n - k^{n+1}]|a_k||z|^{k+1} - \zeta|-1|^n + \sum_{k=1}^{\infty}[(1+\zeta)k^n + k^{n+1}]|a_k||z|^{k+1}$$

$$= 2[(1-\zeta) - \sum_{k=1}^{\infty}(k^n(\zeta + k))]|a_k| \leq 0.$$

From the last inequality, we obtain the condition (72) of the theorem. □

Corollary 3.6. *Let $g \in \mathcal{K}_0^n(\zeta)$, then*

$$|a_k| \leq \frac{1-\zeta}{k^n(\zeta + k)}, k \geq 1. \tag{73}$$

Equality is obtained for

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1-\zeta}{k^n(\zeta + k)}. \tag{74}$$

Theorem 3.8. *Let the function of the form* (1) *be in* $\mathcal{K}_0^n(\zeta)$, *then for* $0 < |z| = r < 1$,

$$\frac{1}{r} - \frac{1-\zeta}{k^n(\zeta+k)}r \leq |f(z)| \leq \frac{1}{r} + \frac{1-\zeta}{k^n(\zeta+k)}r \tag{75}$$

and

$$\frac{1}{r^2} - \frac{k(1-\zeta)}{k^n(\zeta+k)}r^{k-1} \leq |f^{'}(z)| \leq \frac{1}{r^2} + \frac{k(1-\zeta)}{k^n(\zeta+k)}r^{k-1}. \tag{76}$$

Equality in (75) *and* (76) *are obtained for the function*

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1-\zeta}{k^n(\zeta+k)}. \tag{77}$$

Proof. Since $f \in \mathcal{K}_0^n(\zeta)$, then from (72), we have that

$$\sum_{k=1}^{\infty} |a_k| \leq \frac{1-\zeta}{k^n(\zeta+k)}. \tag{78}$$

For $0 < |z| = r < 1$, then

$$|f(z)| \leq \left|\frac{1}{z}\right| + \left|\sum_{k=1}^{\infty} a_k z^k\right|$$

$$\leq \frac{1}{r} + r\sum_{k=1}^{\infty} |a_k|$$

$$\leq \frac{1}{r} + \frac{1-\zeta}{k^n(\zeta+k)}r$$

and

$$|f(z)| \geq \left|\frac{1}{z}\right| - \left|\sum_{k=1}^{\infty} a_k z^k\right|$$

$$\geq \frac{1}{r} - r\sum_{k=1}^{\infty} |a_k|$$

$$\geq \frac{1}{r} - \frac{1-\zeta}{k^n(\zeta+k)}r.$$

Also

$$|f^{'}(z)| \leq \frac{1}{|z|^2} + \left|\sum_{k=1}^{\infty} k a_k z^{k-1}\right|$$

$$|f^{'}(z)| \leq \frac{1}{r^2} + r^{k-1}\sum_{k=1}^{\infty} k|a_k|$$

$$|f^{'}(z)| \leq \frac{1}{r^2} + \frac{k(1-\zeta)}{k^n(\zeta+k)}r^{k-1}$$

and

$$|f^{'}(z)| \geq \frac{1}{|z|^2} - \left|\sum_{k=1}^{\infty} ka_k z^{k-1}\right|$$

$$|f^{'}(z)| \geq \frac{1}{r^2} - r^{k-1}\sum_{k=1}^{\infty} k|a_k|$$

$$|f^{'}(z)| \geq \frac{1}{r^2} - \frac{k(1-\zeta)}{k^n(\zeta+k)} r^{k-1}.$$

□

4. Conclusion

In this work we determine the properties which includes inclusion, integral representation, closure under an integral operator, sufficient condition, coefficient inequality, growth and distortion.

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