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Exploring new trivariate copulas through the C-lifting method

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Abstract: Copulas are special functions that have become essential in the modeling of complex dependence structures between random variables. This is especially true for the three-dimensional case, for which they have attracted much applied interest in recent years. This article innovates on the topic by creating new trivariate copulas derived from the C-lifting method and the recently developed bivariate one-parameter power variable copulas. They have the features of being singular in their expression while keeping generality, with the implication of one or two intermediary functions. In particular, some of them use the incomplete and complete beta functions, which remain original tools in the construction of copulas. With the aim of introducing a totally independent tuning parameter, a strategy mixing the C-lifting method, bivariate one-parameter power variable copulas, and the Farlie-Gumbel-Morgenstern copula is also developed. The trivariate copulas obtained offer new theoretical and practical options for dependence modeling in three dimensions.

Key words: Trivariate copulas, dependence modeling, one-parameter power variable copulas, integral calculus

1. Introduction

Understanding the dependences between multiple random variables is fundamental to many disciplines, including finance, insurance, engineering, meteorology, hydrology, telecommunications, biology, and medicine. However, traditional approaches, which are mainly centered around the correlation measure concept, often fail to capture nuanced dependence contexts. In particular, they are unable to reflect various tail and asymmetric aspects. The theory of copulas offers powerful mathematical tools for this purpose; it has attracted much attention for its ability to model complex dependences while retaining the flexibility to incorporate various marginal distributions. The key references on this topic include [21], [22], [18], [15], [9], and [17].

Although the bivariate situation is well understood, truly original copulas with higher dimensions are still rare. This is especially true in the trivariate case. For this reason, in recent years, copula-based trivariate modeling has evolved significantly, giving rise to various methodologies aimed at improving its accuracy and applicability. See [20], [8], [24], [14], [1], [11], [16], [19], [13], [2], and [26]. One such advance is the C-lifting method established in [20] and refined in [8], which provides an elegant framework for extending bivariate copulas to three dimensions. This method is based on partial differentiations of previously chosen bivariate copulas and on integration. It avoids the need to directly specify a complex trivariate copula function, making it an adaptable tool for capturing dependences. This claim is supported in [8], [12] and [6].

On the other hand, the one-parameter power variable copula family is a recent innovation that presents an interesting alternative to existing copula families. Its particularity lies in its simplicity and its ability to modulate the strength of dependence and the asymmetry of the tail. The paragon of this family is the famous

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Gumbel-Barnett copula. In the bivariate case, it is generally formulated as

$$C(x,y)=xy\exp[-\alpha\log(x)\log(y)],\quad (x,y)\in[0,1]^2, \tag{1}$$

with $\alpha\in[0,1]$. It can also be expressed in the following one-parameter power variable form:

$$C(x,y)=xy^{1-\alpha\log(x)},\quad (x,y)\in[0,1]^2.$$

In [3], bivariate copulas of one-parameter power variable nature are examined in the general form $C(x,y)=xy^{\phi(x)}$, where $\phi(x)$ denotes a function depending only on x (no y). Some specific assumptions on $\phi(x)$ making $C(x,y)$ a valid copula have been identified. With this, a plethora of one-parameter variable-power bivariate copulas have been added to the body of knowledge. Recent developments on this topic can also be found in [4] and [5].

The first contribution of this article lies in the introduction of a novel approach to trivariate copula modeling that exploits the potential of the C-lifting method and the bivariate one-parameter power variable copulas as introduced in [3]. The obtained copulas are of interest for the following reasons:

(i) They are of moderate complexity, original in expression, and general; they may depend on two intermediate tuning functions. Thus, a long list of trivariate copulas can be generated. Among the possibilities, we exhibit some of them involving the incomplete beta function, which remains an innovative tool in this context.

(ii) Thanks to the aspects described in (i), the introduced copulas offer enhanced flexibility in capturing intricate trivariate dependences while maintaining ease of interpretation and calibration.

In a second part, on the same mathematical foundations, a more technical and constructive strategy is developed. It consists of introducing a tuning parameter that is completely independent of the intermediate functions. More precisely, it is based on the combination of the C-lifting method, one-parameter power variable bivariate copulas, and the Farlie-Gumbel-Morgenstern (FGM) copula. The high level of generality of the introduced copulas offers an exhaustive collection of new trivariate dependence models. Their potential applicability extends to a wide range of real-world scenarios.

The remainder of this article is organized as follows: Section 2 presents the necessary mathematical materials to comprehend our approach, including the notions of multivariate copulas, the C-lifting method, and the considered one-parameter power variable copula family. Section 3 is devoted to our first contributions, exemplified by numerous concrete examples of trivariate and bivariate copulas. One-parameter extensions exploiting the FGM copula are developed in Section 4. Finally, Section 5 concludes the article by summarizing the findings and discussing future research directions.

2. Materials

In this section, we present the notion of multivariate copulas, the C-lifting method, and the foundation of bivariate one-parameter power variable copulas.

2.1. Multivariate copulas

To begin, let us precise the definition of the considered multivariate copulas, which is intentionally restricted to the absolutely continuous case.

Definition 2.1. Let us consider an integer n with $n \geq 2$, which represents the dimension of interest. In dimension n, the function $C(u_1, \ldots, u_n)$, $(u_1, \ldots, u_n) \in [0,1]^n$, is a multivariate absolutely continuous copula if and only if, for any $(u_1, \ldots, u_n) \in [0,1]^n$, $i = 1, \ldots, n$, the following conditions are fulfilled:

(A1): $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = 0$,

(A2): $C(1, \ldots, 1, u, 1, \ldots, 1) = u$ for any $u \in [0,1]$, and this when u moving in each of the components,

(A3): $\partial_{u_1,\ldots,u_n} C(u_1, \ldots, u_n) \geq 0$, where $\partial_{u_1,\ldots,u_n} = \partial^n/(\partial u_1 \ldots \partial u_n)$.

For additional details on this notion, see [18]. In the rest of the article, the phrase "absolutely continuous" is omitted from the text moving forward to make it lighter. Also, we mainly focus on the cases $n = 2$ and $n = 3$, corresponding to the bivariate and trivariate cases, respectively. In our coming developments, we never directly use the conditions **(A1)**, **(A2)**, and **(A3)**, but rather a direct method ensuring to construct a trivariate copula based on bivariate copulas, called the C-lifting method, which is presented in the next part.

2.2. C-lifting method

The considered C-lifting method is taken from [8, Proposition 3.2] and [12, Equation (2.9)] with "$C_t = C$". It can be described as a way of obtaining trivariate copulas, starting with two suitable bivariate copulas. It is described in the proposition below.

Proposition 2.1. *[12, Equation (2.9)] Let $C_1(x, y)$, $C_2(x, y)$ and $C_3(x, y)$ be three bivariate copulas. Then the following trivariate integral function is a valid copula:*

$$C(u, v, w) = \int_0^w C_1 \left[\partial_y C_2(u, y)\mid_{y=t}, \partial_y C_3(v, y)\mid_{y=t}\right] dt, \quad (u, v, w) \in [0,1]^3, \tag{2}$$

where $\partial_y = \partial/(\partial y)$.

The trivariate copula obtained by the C-lifting method can also be expressed as an incomplete product of copulas and denoted as $C(u, v, w) = (C_2 \star_{C_1} C_3)(u, v, w)$. The main interests of this method are (i) to produce trivariate copulas based on bivariate copulas without the need to check **(A1)**, **(A2)**, and **(A3)**, and (ii) to obtain original expressions for these trivariate copulas, which remain of interest in dependence modeling in view of the diversity of the observed phenomena. These aspects have been emphasized in diverse contexts in [8], [12] and [6]. A limitation of the C-lifting method is that $C_1(x, y)$, $C_2(x, y)$ and $C_3(x, y)$ must be chosen as "not too complex" to make the integral in Equation (2) manageable. This leads to the main idea of the article: this limitation is perfectly overcome by the use of bivariate one-parameter power variable copulas, as developed in the next part.

2.3. Bivariate one-parameter power variable copulas

First of all, based on a generic function $\phi(x)$, $x \in [0,1]$, we formulate the following key assumptions:

(D1): $\phi(1) = 1$ (or $\lim_{x\to 1} \phi(x) = 1$),

(D2): $\lim_{x\to 0} \phi(x) = \ell$ with $\ell \in [1, +\infty) \cup \{+\infty\}$,

(D3): $\phi(x)$ differentiable with $\phi'(x) \leq 0$,

(D4): $x\phi'(x)+\phi(x)\geq 0$.

These assumptions are satisfied by functions of diverse natures. In particular, according to [3, Lemma 1], the following functions satisfy **D1**, **D2**, **D3**, and **D4**:

- $\phi(x)=1+\beta(1-x^{\alpha})^{\gamma}$, for $\alpha>0$, $\gamma\geq 1$ and $\beta\in[0,1/(\alpha\gamma)]$.
- $\phi(x)=\exp[\beta(1-x^{\alpha})]$, for $\alpha>0$ and $\beta\in[0,1/\alpha]$.
- $\phi(x)=1+\lambda\sin[\beta(1-x^{\alpha})]$, for $\beta\in[0,\pi/2]$, $\alpha>0$, and $\lambda\in[0,1/(\alpha\beta)]$.
- $\phi(x)=1-\beta\log(x)$, for $\beta\in[0,1]$.
- $\phi(x)=\beta(x^{-\alpha}-1)+1$, for $\alpha\in(0,1]$ and $\beta\in[0,1/\alpha]$.

Based on these assumptions, the next result serves as the foundation of the bivariate one-parameter power variable copulas. It is not new; it has been established in [3].

Theorem 2.1. *[3, Theorem 1]. Let $\phi(x)$, $x\in[0,1]$ be a function satisfying the assumptions* **D1***,* **D2***,* **D3***, and* **D4***. Then the following bivariate function is a valid copula:*

$$C(x,y)=xy^{\phi(x)},\quad (x,y)\in[0,1]^2, \tag{3}$$

with the convention that 0^0 exists, say $0^0=1$ (see [25]).

Furthermore, based on [3, Proposition 1], using the x-and y-flipping methods (see [7]), the following bivariate copulas are derived from $C(x,y)=xy^{\phi(x)}$:

1. the x-flipping version of $C(x,y)$ is the copula given by

$$C^{x\text{-flip}}(x,y)=y-(1-x)y^{\phi(1-x)},\quad (x,y)\in[0,1]^2. \tag{4}$$

2. the y-flipping version of $C(x,y)$ is the copula given by

$$C^{y\text{-flip}}(x,y)=x\left[1-(1-y)^{\phi(x)}\right],\quad (x,y)\in[0,1]^2. \tag{5}$$

Based on the above material, we are in a position to present the findings of the article.

3. New trivariate copulas

The next proposition combines the main approaches described in the previous section to elaborate original trivariate copulas beyond the standard ones of the literature.

Proposition 3.1. *Let $f(x)$ and $g(x)$, $x\in[0,1]$, be two functions satisfying the assumptions* **D1***,* **D2***,* **D3***, and* **D4***. Then the following trivariate functions are valid copulas:*

- *First new trivariate copula:*

$$C(u,v,w)=\frac{uvf(u)g(v)}{f(u)+g(v)-1}w^{f(u)+g(v)-1},\quad (u,v,w)\in[0,1]^3. \tag{6}$$

- *Second new trivariate copula:*

$$C(u,v,w) = uvf(u)g(v)B_w\left[f(u),g(v)\right], \quad (u,v,w)\in[0,1]^3, \tag{7}$$

where $B_z(x,y)=\int_0^z t^{x-1}(1-t)^{y-1}dt$ is the incomplete beta function.

- *Third new trivariate copula:*

$$C(u,v,w) = \frac{uvf(u)g(v)}{f(u)+g(v)-1}\left[1-(1-w)^{f(u)+g(v)-1}\right], \quad (u,v,w)\in[0,1]^3.$$

- *Fourth new trivariate copula:*

$$C(u,v,w) = uw^{f(u)} - \frac{u(1-v)f(u)g(1-v)}{f(u)+g(1-v)-1}w^{f(u)+g(1-v)-1}, \quad (u,v,w)\in[0,1]^3.$$

- *Fifth new trivariate copula:*

$$C(u,v,w) = u\left[1-(1-w)^{f(u)}\right] - u(1-v)f(u)g(1-v)B_w[g(1-v),f(u)],$$
$$(u,v,w)\in[0,1]^3.$$

- *Sixth new trivariate copula:*

$$\begin{aligned}C(u,v,w) &= w-(1-u)w^{f(1-u)}-(1-v)w^{g(1-v)}\\&\quad+\frac{(1-u)(1-v)f(1-u)g(1-v)}{f(1-u)+g(1-v)-1}w^{f(1-u)+g(1-v)-1}, \quad (u,v,w)\in[0,1]^3.\end{aligned}$$

Proof. The proof is based on the C-lifting method applied to $C_1(x,y)=xy$ (the independence copula), and $C_1(x,y)$ and $C_2(x,y)$ chosen among the copulas given in Equations (3), (4) and (5), with $\phi(x)=f(x)$ or $\phi(x)=g(x)$. Theorem 2.1 is constantly applied in all the coming results.

- First new trivariate copula: Let us consider the following copulas: $C_1(x,y)=xy$, $C_2(x,y)=xy^{f(x)}$ and $C_3(x,y)=xy^{g(x)}$ (see Equation (3)). Then we have $\partial_y C_2(x,y)=xf(x)y^{f(x)-1}$ and $\partial_y C_3(x,y)=xg(x)y^{g(x)-1}$. The C-lifting method outlined in Equation (2) gives the following trivariate copula:

$$\begin{aligned}C(u,v,w) &= \int_0^w C_1\left[\partial_y C_2(u,y)\mid_{y=t}, \partial_y C_3(v,y)\mid_{y=t}\right]dt\\&= \int_0^w uf(u)t^{f(u)-1}vg(v)t^{g(v)-1}dt\\&= uvf(u)g(v)\int_0^w t^{f(u)+g(v)-2}dt\\&= \frac{uvf(u)g(v)}{f(u)+g(v)-1}w^{f(u)+g(v)-1}.\end{aligned}$$

The desired expression is obtained.

- Second new trivariate copula: Let us consider the following copulas: $C_1(x,y)=xy$, $C_2(x,y)=xy^{f(x)}$ (see Equation (3)) and $C_3(x,y)=x\left[1-(1-y)^{g(x)}\right]$ (see Equation (5)). Then we have $\partial_y C_2(x,y)=xf(x)y^{f(x)-1}$ and $\partial_y C_3(x,y)=xg(x)(1-y)^{g(x)-1}$. The trivariate copula obtained by the C-lifting method described in Equation (2) is as follows:

$$
\begin{aligned}
C(u,v,w) &= \int_0^w C_1\left[\partial_y C_2(u,y)\mid_{y=t}, \partial_y C_3(v,y)\mid_{y=t}\right] dt \\
&= \int_0^w uf(u)t^{f(u)-1} vg(v)(1-t)^{g(v)-1} dt \\
&= uvf(u)g(v)\int_0^w t^{f(u)-1}(1-t)^{g(v)-1} dt \\
&= uvf(u)g(v)B_w\left(f(u),g(v)\right).
\end{aligned}
$$

The stated expression is clarified.

- Third new trivariate copula: Let us consider the following copulas: $C_1(x,y)=xy$, $C_2(x,y)=x\left[1-(1-y)^{f(x)}\right]$ and $C_3(x,y)=x\left[1-(1-y)^{g(x)}\right]$ (see Equation (5)). Then we have $\partial_y C_2(x,y)=xf(x)(1-y)^{f(x)-1}$ and $\partial_y C_3(x,y)=xg(x)(1-y)^{g(x)-1}$. The C-lifting method described in Equation (2) gives the following trivariate copula:

$$
\begin{aligned}
C(u,v,w) &= \int_0^w C_1\left[\partial_y C_2(u,y)\mid_{y=t}, \partial_y C_3(v,y)\mid_{y=t}\right] dt \\
&= \int_0^w uf(u)(1-t)^{f(u)-1} vg(v)(1-t)^{g(v)-1} dt \\
&= uvf(u)g(v)\int_0^w (1-t)^{f(u)+g(v)-2} dt \\
&= \frac{uvf(u)g(v)}{f(u)+g(v)-1}\left[1-(1-w)^{f(u)+g(v)-1}\right].
\end{aligned}
$$

The desired result is obtained.

- Fourth new trivariate copula: Let us consider the following copulas: $C_1(x,y)=xy$, $C_2(x,y)=xy^{f(x)}$ (see Equation (3)) and $C_3(x,y)=y-(1-x)y^{g(1-x)}$ (see Equation (4)). Then we have $\partial_y C_2(x,y)=xf(x)y^{f(x)-1}$ and $\partial_y C_3(x,y)=1-(1-x)g(1-x)y^{g(1-x)-1}$. The trivariate copula obtained by the C-lifting method outlined in Equation (2) is as follows:

$$
\begin{aligned}
C(u,v,w) &= \int_0^w C_1\left[\partial_y C_2(u,y)\mid_{y=t}, \partial_y C_3(v,y)\mid_{y=t}\right] dt \\
&= \int_0^w uf(u)t^{f(u)-1}\left[1-(1-v)g(1-v)t^{g(1-v)-1}\right] dt \\
&= u\int_0^w f(u)t^{f(u)-1}dt - u(1-v)f(u)g(1-v)\int_0^w t^{f(u)+g(1-v)-2}dt \\
&= uw^{f(u)} - \frac{u(1-v)f(u)g(1-v)}{f(u)+g(1-v)-1}w^{f(u)+g(1-v)-1}.
\end{aligned}
$$

We get the stated expression.

- Fifth new trivariate copula: Let us consider the following copulas: $C_1(x,y) = xy$, $C_2(x,y) = x\left[1-(1-y)^{f(x)}\right]$ (see Equation (5)) and $C_3(x,y) = y-(1-x)y^{g(1-x)}$ (see Equation (4)). Then we have $\partial_y C_2(x,y) = xf(x)(1-y)^{f(x)-1}$ and $\partial_y C_3(x,y) = 1-(1-x)g(1-x)y^{g(1-x)-1}$. The C-lifting method described in Equation (2) gives the following trivariate copula:

$$\begin{aligned}C(u,v,w) &= \int_0^w C_1\left[\partial_y C_2(u,y)\mid_{y=t}, \partial_y C_3(v,y)\mid_{y=t}\right] dt \\ &= \int_0^w uf(u)(1-t)^{f(u)-1}\left[1-(1-v)g(1-v)t^{g(1-v)-1}\right] dt \\ &= u\int_0^w f(u)(1-t)^{f(u)-1}dt - u(1-v)f(u)g(1-v)\int_0^w t^{g(1-v)-1}(1-t)^{f(u)-1}dt \\ &= u\left[1-(1-w)^{f(u)}\right] - u(1-v)f(u)g(1-v)B_w[g(1-v),f(u)].\end{aligned}$$

The desired copula is obtained.

- Sixth new trivariate copula: Let us consider the following copulas: $C_1(x,y) = xy$, $C_2(x,y) = y-(1-x)y^{f(1-x)}$ and $C_3(x,y) = y-(1-x)y^{g(1-x)}$ (see Equation (4)). Then we have $\partial_y C_2(x,y) = 1-(1-x)f(1-x)y^{f(1-x)-1}$ and $\partial_y C_3(x,y) = 1-(1-x)g(1-x)y^{g(1-x)-1}$. The trivariate copula obtained by the C-lifting method described in Equation (2) is as follows:

$$\begin{aligned}C(u,v,w) &= \int_0^w C_1\left[\partial_y C_2(u,y)\mid_{y=t}, \partial_y C_3(v,y)\mid_{y=t}\right] dt \\ &= \int_0^w \left[1-(1-u)f(1-u)t^{f(1-u)-1}\right]\left[1-(1-v)g(1-v)t^{g(1-v)-1}\right] dt \\ &= \int_0^w dt - (1-u)\int_0^w f(1-u)t^{f(1-u)-1}dt - (1-v)\int_0^w g(1-v)t^{g(1-v)-1}dt \\ &+ (1-u)(1-v)f(1-u)g(1-v)\int_0^w t^{f(1-u)+g(1-v)-2}dt \\ &= w-(1-u)w^{f(1-u)} - (1-v)w^{g(1-v)} \\ &+ \frac{(1-u)(1-v)f(1-u)g(1-v)}{f(1-u)+g(1-v)-1}w^{f(1-u)+g(1-v)-1}.\end{aligned}$$

This ends the proof of the proposition. □

Based on the trivariate copulas presented in Proposition 3.1, other copulas can be derived by interchanging the variables u, v and w, as well as $f(x)$ and $g(x)$. Since the assumptions made on $f(x)$ and $g(x)$ are large, we can create a wide panel of new trivariate copulas. For example, let us set

$$f(x) = 1-\beta\log(x), \quad g(x) = 1-\gamma\log(x),$$

where $\beta \in [0,1]$ and $\gamma \in [0,1]$. Then, as it is exemplified in [3, Lemma 1], $f(x)$ and $g(x)$ satisfy **D1**, **D2**, **D3**, and **D4**. The findings of Proposition 3.1 give the following two-parameter trivariate copulas:

- First example of two-parameter copula:

$$C(u,v,w)=\frac{uv[1-\beta\log(u)][1-\gamma\log(v)]}{1-\beta\log(u)-\gamma\log(v)}w^{1-\beta\log(u)-\gamma\log(v)},\quad (u,v,w)\in[0,1]^3.$$

- Second example of two-parameter copula:

$$C(u,v,w)=uv[1-\beta\log(u)][1-\gamma\log(v)]B_w\left[1-\beta\log(u),1-\gamma\log(v)\right],$$
$$(u,v,w)\in[0,1]^3.$$

- Third example of two-parameter copula:

$$C(u,v,w)=\frac{uv[1-\beta\log(u)][1-\gamma\log(v)]}{1-\beta\log(u)-\gamma\log(v)}\left[1-(1-w)^{1-\beta\log(u)-\gamma\log(v)}\right],$$
$$(u,v,w)\in[0,1]^3.$$

- Fourth example of two-parameter copula:

$$C(u,v,w)=uw^{1-\beta\log(u)}-\frac{u(1-v)[1-\beta\log(u)][1-\gamma\log(1-v)]}{1-\beta\log(u)-\gamma\log(1-v)}w^{1-\beta\log(u)-\gamma\log(1-v)},$$
$$(u,v,w)\in[0,1]^3.$$

- Fifth example of two-parameter copula:

$$C(u,v,w)=u\left[1-(1-w)^{1-\beta\log(u)}\right]-u(1-v)[1-\beta\log(u)][1-\gamma\log(1-v)]\times$$
$$B_w[1-\gamma\log(1-v)],1-\beta\log(u)],\quad (u,v,w)\in[0,1]^3.$$

- Sixth example of two-parameter copula:

$$C(u,v,w)=w-(1-u)w^{1-\beta\log(1-u)}-(1-v)w^{1-\gamma\log(1-v)}$$
$$+\frac{(1-u)(1-v)[1-\beta\log(1-u)][1-\gamma\log(1-v)]}{1-\beta\log(1-u)-\gamma\log(1-v)}w^{1-\beta\log(1-u)-\gamma\log(1-v)},$$
$$(u,v,w)\in[0,1]^3.$$

We can list so many more examples by combining Proposition 3.1 and [3, Lemma 1].

Of course, new trivariate distributions can be derived from these copulas. More precisely, for three univariate cumulative distribution functions, say $F(x)$, $G(x)$, and $H(x)$, the following trivariate function defines a valid cumulative distribution function:

$$K(x,y,z)=C[F(x),G(y),H(z)],\quad (x,y,z)\in\mathbb{R}^3.$$

In this setting, we can define a trivariate random vector (X,Y,Z) on a probability space, say $(\Omega,\mathcal{F},P)$, such that $P(X\le x,Y\le y,Z\le z)=K(x,y,z)$, $(x,y,z)\in\mathbb{R}^3$. With this simple scheme, a wide variety of trivariate random vectors and distributions can be introduced. For motivated choices of univariate cumulative lifetime distribution functions $F(x)$, $G(x)$, and $H(x)$, we may refer to [23]. Thanks to the originality of the proposed copulas, we thus offer alternatives to the trivariate distributions described in [20], [8], [24], [14], [1], [11], [16], [19], [13], and [26], among others.

Another aspect is that we can derive bivariate copulas based on trivariate copulas by putting one variable at 1 and keeping the two others as main variables. In particular, based on Proposition 3.1, by setting $w = 1$, the following bivariate copulas are derived:

- General ratio bivariate copula:

$$C(u,v) = \frac{uvf(u)g(v)}{f(u)+g(v)-1}, \quad (u,v) \in [0,1]^2.$$

It generalizes the ratio bivariate copula outlined in [3, Proposition 4] thanks to the presence of the function $g(x)$.

- Beta bivariate copula:

$$C(u,v) = uvf(u)g(v)B\left[f(u),g(v)\right], \quad (u,v) \in [0,1]^2,$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = B_1(x,y)$ is the (complete) beta function. To the best of our knowledge, this is the first time that this bivariate copula has been communicated.

The other bivariate copulas that can be obtained from Proposition 3.1 by taking $w = 1$ come from the x- or y-flipping or survival transformations of the two copulas mentioned above.

In order to concretize more of the findings, if we consider the following simple functions:

$$f(x) = 1 - \beta\log(x), \quad g(x) = 1 - \gamma\log(x),$$

where $\beta \in [0,1]$ and $\gamma \in [0,1]$, then the two previous two-parameter bivariate copulas become

$$C(u,v) = \frac{uv[1-\beta\log(u)][1-\gamma\log(v)]}{1-\beta\log(u)-\gamma\log(v)}, \quad (u,v) \in [0,1]^2,$$

which also appears in [10, Example 2.6], and

$$C(u,v) = uv[1-\beta\log(u)][1-\gamma\log(v)]B\left[1-\beta\log(u), 1-\gamma\log(v)\right], \quad (u,v) \in [0,1]^2,$$

respectively. Let us call this last bivariate copula the beta copula. In order to visualize its validity and flexibility in terms of shape, Figures 1 and 2 display it for various values of β and γ.

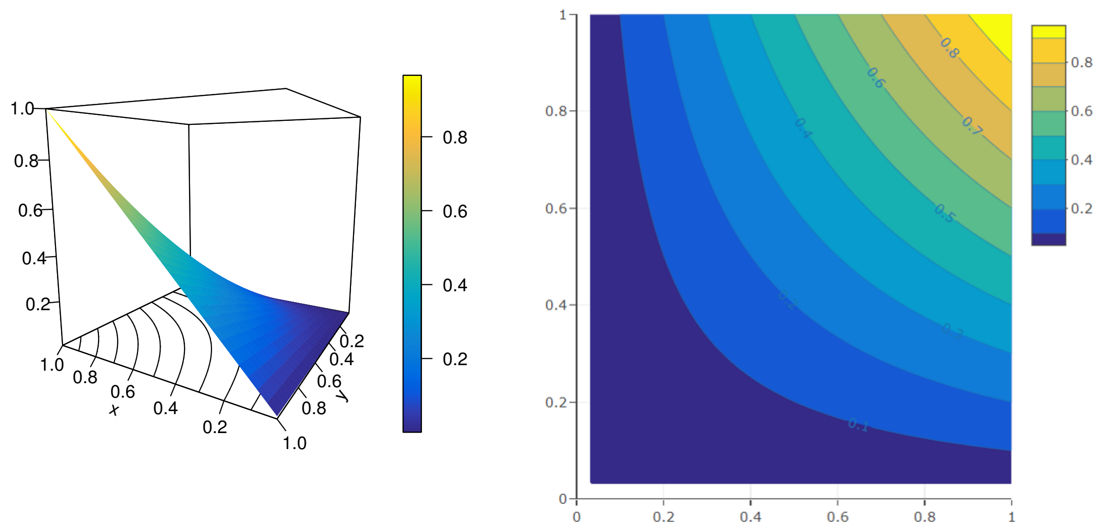

Figure 1. Plots for the beta copula with $\beta = \gamma = 0.1$: standard (left) and intensity-contour (right)

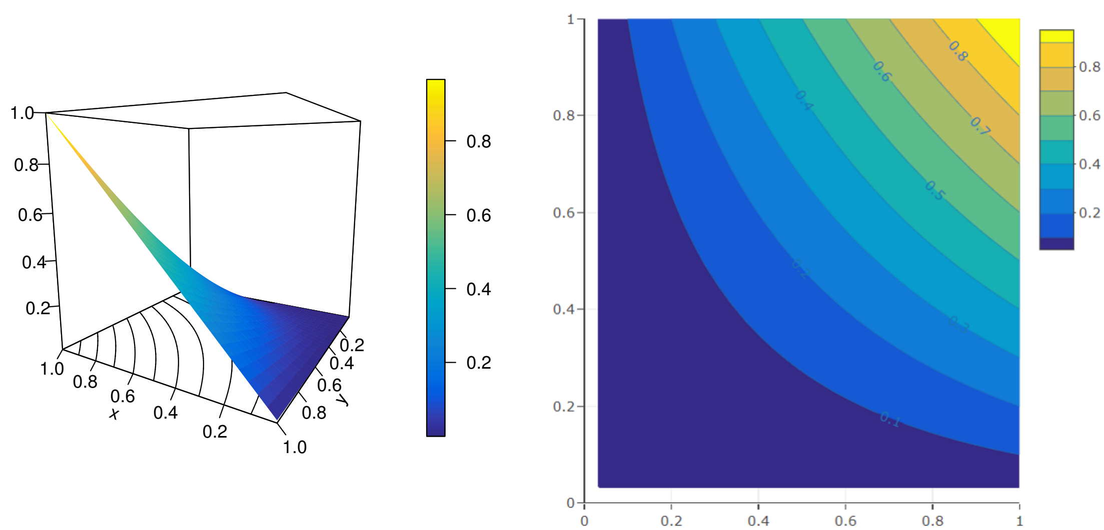

Figure 2. Plots for the beta copula with $\beta = \gamma = 0.9$: standard (left) and intensity-contour (right)

From these figures, the typical form of a valid copula is obtained, with varying intensity and contour depending on the values of β and γ. The beta copula requires a more deep study that we leave for future work.

4. One-parameter extensions

In this section, based on the above methodology, we investigate some strategies to introduce parameters beyond those possibly introduced in the intermediary functions $f(x)$ and $g(x)$. Based on the C-lifting method described in Equation (2), an idea is to employ a one-parameter copula for $C_1(x, y)$, or for $\{C_2(x, y)$ or $C_3(x, y)\}$. This strategy has a chance of success if simple copula configurations are considered. Here we focus on the FGM copula indicated as

$$C(x, y) = xy[1 + \lambda(1 - x)(1 - y)], \quad (x, y) \in [0, 1]^2, \tag{8}$$

with $\lambda \in [-1, 1]$. We mention that the FGM copula is known to be reduced to the independence copula with $\lambda = 0$, and it is characterized by its ability to capture both positive and negative tail dependences, making it a versatile tool in applications, especially in risk assessment and financial modeling (see [18]).

The next proposition presents some tractable trivariate copulas with only one general function $f(x)$ and a tuning parameter independent of this function.

Proposition 4.1. *Let $f(x)$, $x \in [0, 1]$, be a function satisfying the assumptions* **D1**, **D2**, **D3**, *and* **D4**, *and $\lambda \in [-1, 1]$. Then the following one-additional-parameter trivariate functions are valid copulas:*

- *First new one-additional-parameter trivariate copula:*

$$C(u, v, w) = uvw^{f(u)} \left[1 + \lambda(1 - v)\frac{(1 - 2w)f(u) + 1}{f(u) + 1}\right], \quad (u, v, w) \in [0, 1]^3.$$

- *Second new one-additional-parameter trivariate copula:*

$$\begin{aligned} C(u, v, w) &= uv[1 - (1 - w)^{f(u)}] \\ &\quad + \lambda uv(1 - v)\left\{\frac{f(u) - 1}{f(u) + 1} + (1 - w)^{f(u)}\frac{[1 - 2(1 - w)]f(u) + 1}{f(u) + 1}\right\}, \\ &\quad (u, v, w) \in [0, 1]^3. \end{aligned}$$

Proof. The proof is based on the C-lifting method applied to $C_1(x, y) = xy$ (the independence copula), and $C_1(x, y)$ and $C_2(x, y)$ chosen among the copulas given in Equations (3), (5) and (8), with $\phi(x) = f(x)$. Theorem 2.1 is constantly applied in all the coming results.

- First new one-additional-parameter trivariate copula: Let us consider the following copulas: $C_1(x, y) = xy$, $C_2(x, y) = xy^{f(x)}$ (see Equation (3)) and $C_3(x, y) = xy[1 + \lambda(1 - x)(1 - y)]$ (see Equation (8)). Then we have $\partial_y C_2(x, y) = xf(x)y^{f(x)-1}$ and $\partial_y C_3(x, y) = x + \lambda x(1 - x)(1 - 2y)$. The C-lifting method outlined in Equation (2) gives the following trivariate copula:

$$\begin{aligned} C(u, v, w) &= \int_0^w C_1 \left[\partial_y C_2(u, y) \mid_{y=t}, \partial_y C_3(v, y) \mid_{y=t}\right] dt \\ &= \int_0^w uf(u)t^{f(u)-1} \left[v + \lambda v(1 - v)(1 - 2t)\right] dt \\ &= uv \int_0^w f(u)t^{f(u)-1} dt + \lambda uv(1 - v) \left[\int_0^w f(u)t^{f(u)-1} dt - 2\int_0^w f(u)t^{f(u)} dt\right] \\ &= uvw^{f(u)} + \lambda uv(1 - v)\left[w^{f(u)} - 2\frac{f(u)}{f(u) + 1}w^{f(u)+1}\right] \\ &= uvw^{f(u)}\left[1 + \lambda(1 - v)\frac{(1 - 2w)f(u) + 1}{f(u) + 1}\right]. \end{aligned}$$

The desired expression is obtained.

- Second new one-additional-parameter trivariate copula: Let us consider the following copulas: $C_1(x,y) = xy$, $C_2(x,y) = x\left[1-(1-y)^{f(x)}\right]$ (see Equation (5)) and $C_3(x,y) = xy[1+\lambda(1-x)(1-y)]$ (see Equation (8)). Then we have $\partial_y C_2(x,y) = xf(x)(1-y)^{f(x)-1}$ and $\partial_y C_3(x,y) = x + \lambda x(1-x)(1-2y)$. The trivariate copula deduced from the C-lifting method described in Equation (2) is as follows:

$$\begin{aligned} C(u,v,w) &= \int_0^w C_1\left[\partial_y C_2(u,y)\mid_{y=t}, \partial_y C_3(v,y)\mid_{y=t}\right] dt \\ &= \int_0^w uf(u)(1-t)^{f(u)-1}\left[v+\lambda v(1-v)(1-2t)\right] dt \\ &= uv\int_0^w f(u)(1-t)^{f(u)-1} dt \\ &+ \lambda uv(1-v)\left[\int_0^w f(u)(1-t)^{f(u)-1}dt - 2\int_0^w f(u)t(1-t)^{f(u)-1}dt\right] \\ &= uv[1-(1-w)^{f(u)}] + \lambda uv(1-v)\left[1-(1-w)^{f(u)}-2I\right], \end{aligned}$$

where

$$I = \int_0^w f(u)t(1-t)^{f(u)-1}dt.$$

By applying the change of variables $z = 1-t$, we get

$$\begin{aligned} I &= \int_{1-w}^1 f(u)(1-z)z^{f(u)-1}dz = \int_{1-w}^1 f(u)z^{f(u)-1}dz - \int_{1-w}^1 f(u)z^{f(u)}dz \\ &= 1-(1-w)^{f(u)} - \frac{f(u)}{f(u)+1}\left[1-(1-w)^{f(u)+1}\right]. \end{aligned}$$

Therefore, by putting the equations above together, we obtain

$$\begin{aligned} C(u,v,w) &= uv[1-(1-w)^{f(u)}] \\ &\quad + \lambda uv(1-v)\left\{\frac{f(u)-1}{f(u)+1} + (1-w)^{f(u)}\frac{[1-2(1-w)]f(u)+1}{f(u)+1}\right\}. \end{aligned}$$

The stated expression is obtained.

This ends the proof of the proposition. □

Like for Proposition 3.1, based on the trivariate copulas presented in Proposition 4.1, other copulas can be derived by interchanging the variables u, v, and w. Since the assumptions made on $f(x)$ are flexible, we can create a large panel of new trivariate copulas. As an example, let us set

$$f(x) = 1-\beta\log(x),$$

where $\beta \in [0,1]$. Then the findings of Proposition 4.1 give the following two-parameter trivariate copulas:

- First new one-additional-parameter trivariate copula:

$$C(u,v,w) = uvw^{1-\beta\log(u)}\left[1+\lambda(1-v)\frac{(1-2w)[1-\beta\log(u)]+1}{2-\beta\log(u)}\right], \quad (u,v,w)\in[0,1]^3.$$

- Second new one-additional-parameter trivariate copula:

$$C(u,v,w) = uv[1-(1-w)^{1-\beta\log(u)}]$$
$$+ \lambda uv(1-v)\left\{-\beta\frac{\log(u)}{2-\beta\log(u)} + (1-w)^{1-\beta\log(u)}\frac{[1-2(1-w)][1-\beta\log(u)]+1}{2-\beta\log(u)}\right\},$$
$$(u,v,w)\in[0,1]^3.$$

To the best of our knowledge, such general two-parameter trivariate copulas innovate in the specialized literature on copulas.

The next proposition develops another strategy dealing with two general functions, $f(x)$ and $g(x)$, and a tuning parameter independent of these functions.

Proposition 4.2. *Let $f(x)$ and $g(x)$, $x\in[0,1]$, be two functions satisfying the assumptions* **D1**, **D2**, **D3**, *and* **D4**, *and $\lambda\in[-1,1]$. Then the following one-additional-parameter trivariate functions are valid copulas:*

- *First new one-additional-parameter trivariate copula:*

$$C(u,v,w) = C_\star(u,v,w) + \lambda uvf(u)g(v)\times$$
$$\left\{\frac{1}{f(u)+g(v)-1}w^{f(u)+g(v)-1} - \frac{uf(u)}{2f(u)+g(v)-2}w^{2f(u)+g(v)-2}\right.$$
$$\left.-\frac{vg(v)}{f(u)+2g(v)-2}w^{f(u)+2g(v)-2} + \frac{uvf(u)g(v)}{2f(u)+2g(v)-3}w^{2f(u)+2g(v)-3}\right\},$$
$$(u,v,w)\in[0,1]^3,$$

where $C_\star(u,v,w)$ is the copula described in Equation (6), i.e.,

$$C_\star(u,v,w) = \frac{uvf(u)g(v)}{f(u)+g(v)-1}w^{f(u)+g(v)-1}.$$

- *Second new one-additional-parameter trivariate copula:*

$$C(u,v,w) = C_\triangle(u,v,w) + \lambda uvf(u)g(v)\times$$
$$\Big\{B_w\left[f(u),g(v)\right] - uf(u)B_w\left[2f(u)-1,g(v)\right] - vg(v)B_w\left[f(u),2g(v)-1\right]$$
$$+ uvf(u)g(v)B_w\left[2f(u)-1,2g(v)-1\right]\Big\},\quad (u,v,w)\in[0,1]^3,$$

where $C_\triangle(u,v,w)$ is the copula outlined in Equation (7), i.e.,

$$C_\triangle(u,v,w) = uvf(u)g(v)B_w\left[f(u),g(v)\right].$$

Proof. The proof is based on the C-lifting method applied with $C_1(x,y)$ given in Equation (8), and $C_1(x,y)$ and $C_2(x,y)$ chosen among the copulas presented in Equations (3) and (5), with $\phi(x)=f(x)$ or $\phi(x)=g(x)$. Theorem 2.1 is constantly applied in all the coming results.

- First new one-additional-parameter trivariate copula: Let us consider the following copulas: $C_1(x,y) = xy[1+\lambda(1-x)(1-y)]$ (see Equation (8)), $C_2(x,y) = xy^{f(x)}$ and $C_3(x,y) = xy^{g(x)}$ (see Equation (3)). Then we have $\partial_y C_2(x,y) = xf(x)y^{f(x)-1}$ and $\partial_y C_3(x,y) = xg(x)y^{g(x)-1}$. The C-lifting method described in Equation (2) gives the following trivariate copula:

$$
\begin{aligned}
C(u,v,w) &= \int_0^w C_1\left[\partial_y C_2(u,y)\mid_{y=t}, \partial_y C_3(v,y)\mid_{y=t}\right] dt \\
&= \int_0^w uf(u)t^{f(u)-1}vg(v)t^{g(v)-1}\left\{1+\lambda\left[1-uf(u)t^{f(u)-1}\right]\left[1-vg(v)t^{g(v)-1}\right]\right\} dt \\
&= uvf(u)g(v)\int_0^w t^{f(u)+g(v)-2}dt \\
&+ \lambda uvf(u)g(v)\int_0^w t^{f(u)+g(v)-2}\left[1-uf(u)t^{f(u)-1}\right]\left[1-vg(v)t^{g(v)-1}\right] dt \\
&= \frac{uvf(u)g(v)}{f(u)+g(v)-1}w^{f(u)+g(v)-1} + \lambda uvf(u)g(v)\times \\
&\left\{\int_0^w t^{f(u)+g(v)-2}\left[1-uf(u)t^{f(u)-1}-vg(v)t^{g(v)-1}+uvf(u)g(v)t^{f(u)+g(v)-2}\right] dt\right\} \\
&= C_\star(u,v,w) + \lambda uvf(u)g(v)\times \\
&\left\{\int_0^w t^{f(u)+g(v)-2}dt - uf(u)\int_0^w t^{2f(u)+g(v)-3}dt - vg(v)\int_0^w t^{f(u)+2g(v)-3}dt\right. \\
&\left. + uvf(u)g(v)\int_0^w t^{2f(u)+2g(v)-4}dt\right\} \\
&= C_\star(u,v,w) + \lambda uvf(u)g(v)\times \\
&\left\{\frac{1}{f(u)+g(v)-1}w^{f(u)+g(v)-1} - \frac{uf(u)}{2f(u)+g(v)-2}w^{2f(u)+g(v)-2}\right. \\
&\left. - \frac{vg(v)}{f(u)+2g(v)-2}w^{f(u)+2g(v)-2} + \frac{uvf(u)g(v)}{2f(u)+2g(v)-3}w^{2f(u)+2g(v)-3}\right\}.
\end{aligned}
$$

The desired formula is get.

- Second new one-additional-parameter trivariate copula: Let us consider the following copulas: $C_1(x,y) = xy[1+\lambda(1-x)(1-y)]$ (see Equation (8)), $C_2(x,y) = xy^{f(x)}$ (see Equation (3)) and $C_3(x,y) = x[1-(1-y)^{g(x)}]$ (see Equation (5)). Then we have $\partial_y C_2(x,y) = xf(x)y^{f(x)-1}$ and $\partial_y C_3(x,y) = xg(x)(1-y)^{g(x)-1}$. The

C-lifting method described in Equation (2) gives the following trivariate copula:

$$C(u,v,w) = \int_0^w C_1\left[\partial_y C_2(u,y)\mid_{y=t}, \partial_y C_3(v,y)\mid_{y=t}\right] dt$$

$$= \int_0^w uf(u)t^{f(u)-1}vg(v)(1-t)^{g(v)-1}\left\{1+\lambda\left[1-uf(u)t^{f(u)-1}\right]\left[1-vg(v)(1-t)^{g(v)-1}\right]\right\} dt$$

$$= uvf(u)g(v)\int_0^w t^{f(u)-1}(1-t)^{g(v)-1}dt$$

$$+ \lambda uvf(u)g(v)\int_0^w t^{f(u)-1}(1-t)^{g(v)-1}\left[1-uf(u)t^{f(u)-1}\right]\left[1-vg(v)(1-t)^{g(v)-1}\right] dt$$

$$= uvf(u)g(v)B_w\left[f(u),g(v)\right] + \lambda uvf(u)g(v)\times$$

$$\left\{\int_0^w t^{f(u)-1}(1-t)^{g(v)-1}\left[1-uf(u)t^{f(u)-1}-vg(v)(1-t)^{g(v)-1}\right.\right.$$

$$\left.\left.+uvf(u)g(v)t^{f(u)-1}(1-t)^{g(v)-1}\right]dt\right\}$$

$$= C_{\triangle}(u,v,w) + \lambda uvf(u)g(v)\times$$

$$\left\{\int_0^w t^{f(u)-1}(1-t)^{g(v)-1}dt - uf(u)\int_0^w t^{2f(u)-2}(1-t)^{g(v)-1}dt\right.$$

$$\left.- vg(v)\int_0^w t^{f(u)-1}(1-t)^{2g(v)-2}dt + uvf(u)g(v)\int_0^w t^{2f(u)-2}(1-t)^{2g(v)-2}dt\right\}$$

$$= C_{\triangle}(u,v,w) + \lambda uvf(u)g(v)\times$$

$$\left\{B_w\left[f(u),g(v)\right] - uf(u)B_w\left[2f(u)-1,g(v)\right] - vg(v)B_w\left[f(u),2g(v)-1\right]\right.$$

$$\left.+ uvf(u)g(v)B_w\left[2f(u)-1,2g(v)-1\right]\right\}.$$

The stated result is obtained.

This ends the proof of the proposition. □

The complex expressions of the trivariate copulas presented in Proposition 4.2 are a drawback, but they have the benefit of being very general because we can modulate two functions, $f(x)$ and $g(x)$ (which we can set to the constant 1 if desired), and a parameter, λ, without any connection between the functions and the parameters.

Thus, the copulas in Proposition 4.2 extend the two first copulas in Proposition 3.1 by the modulation of an extra term governed by the parameter λ, going one step further in the development of original and flexible dependence models.

5. Conclusion

In conclusion, the approach presented here, which combines the C-lifting method with one-parameter power variable bivariate copulas, marks an innovative step forward in the construction of trivariate copulas. As the main positive points, the proposed copulas combine generality and originality, introducing a new modeling perspective. This is especially true for those defined with incomplete and complete beta functions, which are singular tools in this context, to the best of our knowledge. Furthermore, the integration of the Farlie-Gumbel-Morgenstern copula and the proposed strategy further improve the versatility of trivariate copulas, providing new possibilities for modeling dependences in three dimensions. This framework, characterized by the introduction of an independent tuning parameter, is very promising in various practical contexts. Work in this direction offers stimulating perspectives for the future of dependence modeling.

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