Asia Mathematika
Volume: 3 Issue: 2 , (2019) Pages: 72 – 75
Available online at **www.asiamath.org**

$\psi\widehat{g}$-homeomorphism in topological spaces

N.Ramya[1*]
[1]Department of Mathematics, Sri Shakthi Institute of Engineering and Technology, Coimbatore,Tamil Nadu,India.

Received: 08 Jun 2019 • **Accepted:** 05 Jul 2019 • **Published Online:** 01 Aug 2019

Abstract: New class of homeomorphisms named as $\psi\widehat{g}$-homeomorphism is explored and elaborated.Few basic properties are inspected.Their relations with some homeomorphisms in topological spaces are studied.

Key words: $\psi\widehat{g}$-(open)closed functions, $\psi\widehat{g}$-irresolute functions, $\psi\widehat{g}$-homeomorphism

1. Introduction

N Levine[6] introduced the concept of generalized closed sets and the class of continuous function using (g open set)semi open sets. Balachandran[3] et al introduced the concept of generalized continuous map in a topological spaces.The concept of generalized homeomorphism was introduced and studied in the year 1991 by Balachandran et all [7]. Recently, as a generalization of closed sets, the notion of $\psi\widehat{g}$-closed sets were introduced and studied by Ramya N. and Parvathi S. [9]. This paper aims to explore and elaborate $\psi\widehat{g}$-homeomorphism and its relation with some existing homeomorphisms in topological spaces. Few of its properties are investigated.

2. Preliminaries

Throughout this paper (X,τ))(or simply X) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A respectively. Let us recall the following definitions which are useful in the sequel.

Definition 2.1. A subset A of a topological space (X,τ) is called $\psi\widehat{g}$ -closed set if $\psi cl(A) \subseteq A$ whenever $A \subseteq U$ and U is $\widehat{g}$-open in (X,τ).

Definition 2.2. A function f: $(X,\tau) \to (X,\sigma)$ is called
(i) semi continuous if $f^{-1}(V)$ is semi closed in (X,τ) for every closed set V of (Y,σ).
(ii) g- continuous if $f^{-1}(V)$ is g- closed in (X,τ) for every closed set V of (Y,σ).
(iii) sg- continuous if $f^{-1}(V)$ is sg- closed in (X,τ) for every closed set V of (Y,σ).
(iv) gs- continuous if $f^{-1}(V)$ is gs- closed in (X,τ) for every closed set V of (Y,σ).
(v) g^*- continuous if $f^{-1}(V)$ is g^*- closed in (X,τ) for every closed set V of (Y,σ).
(vi) ψ- continuous if $f^{-1}(V)$ is g^*- closed in (X,τ) for every closed set V of (Y,σ).
(vii) ψg- continuous if $f^{-1}(V)$ is - closed in (X,τ) for every closed set V of (Y,σ).

*Correspondence: ramyanagaraj144@gmail.com

(viii) $\psi\widehat{g}$-irresolute if $f^{-1}(V)$ is $\psi\widehat{g}$- closed in (X,τ) for every $\psi\widehat{g}$ closed (resp.closed) subset V of (Y,σ).
(ix) $\psi\widehat{g}$-open($\psi\widehat{g}$-closed) if f(V) is $\psi\widehat{g}$-open(resp. $\psi\widehat{g}$-closed)in (X,τ) for every open (resp.closed) subset of (X,τ).

Definition 2.3. A function f: $(X,\tau) \to (X,\sigma)$ is called
(i) If g is open and continuous then it is called as homeomorphism.
(ii) If g is g-continuous and g-open then it is called as g- homeomorphism.
(iii) If g is sg-continuous and sg-open then it is called as sg-homeomorphism.
(iv) If g is gs-continuous and gs-open then it is called as gs- homeomorphism.

3. $\psi\widehat{g}$-homeomorphism

In this section we introduce $\psi\widehat{g}$-homeomorphism in topological spaces

Definition 3.1. A bijection f: $(X,\tau) \to (Y,\sigma)$ is called $\psi\widehat{g}$-homeomorphism if both $\psi\widehat{g}$-continuous and $\psi\widehat{g}$-open.

Example 3.1. *Let $X = \{a,b,c\}$=Y,with $\tau = \{\phi, X, \{b\}\}$ and $\sigma = \{\phi, Y, \{a,b\}\}$.Let f:$(X,\tau) \to (Y,\sigma)$ be defined by f(a)=b,f(b)=c,f(c)=a.Then f is $\psi\widehat{g}$-homeomorphism*

Theorem 3.1. *Every homeomorphism is also a $\psi\widehat{g}$-homeomorphism*

Proof. Assume that f is a homeomorphism. Then f is both continuous and open. Since every continuous function is $\psi\widehat{g}$-continuous and every open set is $\psi\widehat{g}$-open, f is both $\psi\widehat{g}$-continuous and $\psi\widehat{g}$-open. Hence f is $\psi\widehat{g}$-homeomorphism.
The converse of the above theorem need not be true as seen from the following example. ☐

Example 3.2. *Let $X = \{a,b,c\}$=Y, with $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, Y, \{a,b\}\}$.Let f:$(X,\tau) \to (Y,\sigma)$ be defined by f(a)=b,f(b)=c,f(c)=a.Then f is $\psi\widehat{g}$-homeomorphism .Then f is $\psi\widehat{g}$-homeomorphism but not a homeomorphism, since for the $\psi\widehat{g}$-closed set $\{c\}$ in X, $f^{-1}(c) = \{c\}$ which is not closed set in Y.*

Theorem 3.2. *For any bijection f:$(X,\tau) \to (Y,\sigma)$, the following are equivalent: (i) f is $\psi\widehat{g}$-open function,*
(ii) f is a $\psi\widehat{g}$-closed function
(iii) f:$(X,\tau) \to (Y,\sigma)$ is $\psi\widehat{g}$-continuous
***Proof** (i) $\Longrightarrow$ (ii) Let U be any closed set in (X,τ).Then X-U is open in (X,τ) . By (i) f(X-U)=Y-f(U) is $\psi\widehat{g}$-open in (Y,σ) .Therefore f(U) is $\psi\widehat{g}$-closed in (Y,σ) and hence f is $\psi\widehat{g}$-closed function.*
(ii) $\Longrightarrow$ (iii) Let U be any closed subset of (X,τ).Since f is $\psi\widehat{g}$-closed$(f^{-1})^{-1}$ =f(U) is $\psi\widehat{g}$ -closed in (Y,σ) . Hence f^{-1} is $\psi\widehat{g}$-continuous.
(iii) $\Longrightarrow$ (i) Let U be a open subset of (X,τ).By (iii), f(U)= $(f^{-1})^{-1}(U)$ is $\psi\widehat{g}$-open in (Y,σ) . Hence f is $\psi\widehat{g}$-open.

Theorem 3.3. *For any bijection f:$(X,\tau) \to (Y,\sigma)$, the following are equivalent:*
(i) f is $\psi\widehat{g}$ -open function,
(ii) f is a $\psi\widehat{g}$ -homeomorphism
(iii) f is a $\psi\widehat{g}$-closed function

Remark 3.1. *The composition of two $\psi\widehat{g}$–homeomorphism function need not be a $\psi\widehat{g}$-homeomorphism as seen from the following example.*

Example 3.3. *Let $X = \{a, b, c\}$,with $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, X, \{b\}, \{a, b\}\}$, $\mu = \{\phi, X, \{b\}, \{a, c\}\}$. Define f:$(X, \tau) \to (X, \sigma)$ and g:$(X, \sigma) \to (X, \mu)$ are identity function. Then both f and g are $\psi\widehat{g}$-homeomorphism but their composition $g \circ f$: (X, τ) rightarrow(X, μ) is not a $\psi\widehat{g}$-homeomorphism, because for the closed set $\{b\}$ in (X, τ) ,$(g \circ f)(\{ b\}) = \{b\}$, which is not a $\psi\widehat{g}$- -closed set in (X, τ). Therefore $g \circ f$ is not a $\psi\widehat{g}$-closed function and so $g \circ f$ is not a $\psi\widehat{g}$-homeomorphism.*

Definition 3.2. A bijection f:$(X, \tau) \to (X, \sigma)$, is said to be $\psi\widehat{g}^*$-homeomorphism if both f and f^{-1} are $\psi\widehat{g}$-irresolute.

We denote the family of all $\psi\widehat{g}^*$-homeomorphism of a topological space (X, τ) onto itself by $\psi\widehat{g}^*\,\mathrm{h}(X, \tau)$.

Theorem 3.4. *If f:$(X, \tau) \to (Y, \sigma)$ and g:$(Y, \sigma) \to (Z, \eta)$ are $\psi\widehat{g}^*$-homeomorphism, then their composition $g \circ f$:$(X, \tau) \to (Z, \eta)$ is also a $\psi\widehat{g}^*$-homeomorphism.*

Proof. Let U be a $\psi\widehat{g}$-open set in Z. Now, $(g \circ f)^{-1}(U) = (f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where V = $g^{-1}(U)$. By hypothesis, V is $\psi\widehat{g}$-open in Y and since f is a $\psi\widehat{g}^*$-homeomorphism, $f^{-1}(V)$ is $\psi\widehat{g}$-open in X. Therefore, g $\circ$ f is $\psi\widehat{g}$-irresolute. For any $\psi\widehat{g}$-open set G in X, (g $\circ$ f)(G) = g(f(G)) = g(W), where W = f(G). Since f and g are $\psi\widehat{g}^*$ -homeomorphism f(G) is $\psi\widehat{g}$ -open in Y and g(f(G)) is $\psi\widehat{g}$-open in Z. That is (g $\circ$ f)(G) is $\psi\widehat{g}$-open in Z and therefore $(g \circ f)^{-1}$ is $\psi\widehat{g}$ -irresolute. Hence$(g \circ f)^{-1}$ f is a $\psi\widehat{g}^*$-homeomorphism. ☐

Theorem 3.5. *$\psi\widehat{g}^*$-homeomorphism is an equivalence relation in the collection of all topological spaces.*

Theorem 3.6. *If f:$(X, \tau) \to (Y, \sigma)$ is a $\psi\widehat{g}^*$-homeomorphism, then $\psi\widehat{g}$-cl$(f^{-1}(B))$= f^{-1}($\psi\widehat{g}$-cl(B)) for all $B \subseteq Y$.*

Proof. Since f is a $\psi\widehat{g}^*$-homeomorphism, f is $\psi\widehat{g}$-irresolute. Since $\psi\widehat{g}$-cl(f(B)) is a $\psi\widehat{g}$-closed set in Y ,f^{-1}($\psi\widehat{g}$-cl(f(B))) is $\psi\widehat{g}$ -closed in X . As, f^{-1}(B)$\subseteq$ f^{-1}($\psi\widehat{g}$-cl(B)), $\psi\widehat{g}$-cl$(f^{-1}(B)) \subseteq$ $f^{-1}(\psi\widehat{g} - cl(B))$. Since f is a $\psi\widehat{g}^*$ -homeomorphism, f^{-1} is a $\psi\widehat{g}$-irresolute. Since $\psi\widehat{g}$-cl$(f^{-1}$(B)) is $\psi\widehat{g}$-closed in X, $((f^{-1})^{-1}$($\psi\widehat{g}$-cl$(f^{-1}$(B)))=f($\psi\widehat{g}$ -cl$(f^{-1}$(B))) is $\psi\widehat{g}$-closed in Y. As, B$\subseteq$ $((f^{-1})^{-1})$)$(f^{-1}$(B)) $\subseteq$ $((f^{-1})^{-1})$ ($\psi\widehat{g}$ -cl$(f^{-1}(B))$) =f($\psi\widehat{g}$-cl$(f^{-1}(B))$) and so $\psi\widehat{g}$-cl(B)$\subseteq$f($\psi\widehat{g}$-cl$(f^{-1}(B))$). Therefore, f^{-1}($\psi\widehat{g}$-cl(B))$\subseteq$ f^{-1}(f($\psi\widehat{g}$ -cl$(f^{-1}(B))$)) $\subseteq$ $\psi\widehat{g}$-cl$(f^{-1}(B))$ and hence the equality holds. ☐

Theorem 3.7. *If f:$(X, \tau) \to (Y, \sigma)$ is a $\psi\widehat{g}^*$-homeomorphism, then $\psi\widehat{g}$-cl(f(B))=f($\psi\widehat{g}$-cl(B)) for all $B \subseteq X$.*

Proof. Since f:$(X, \tau) \to (Y, \sigma)$ is a $\psi\widehat{g}^*$-homeomorphism, f^{-1}:$(Y, \sigma) \to (X, \tau)$ is also a $\psi\widehat{g}^*$-homeomorphism. Therefore, by theorem 3.6, $\psi\widehat{g}$-cl($((f^{-1})^{-1}$)(B)) =$(f^{-1})^{-1}$) ($\psi\widehat{g}$-cl(B)) for all B $\subseteq$X. i.e.,$\psi\widehat{g}$-cl(f(B)) = f($\psi\widehat{g}$-cl(B)). ☐

Theorem 3.8. *If f:$(X, \tau) \to (Y, \sigma)$ is a $\psi\widehat{g}^*$-homeomorphism, then f($\psi\widehat{g}$-int(B))= $\psi\widehat{g}$-int(f(B)) for all $B \subseteq X$.*

Proof. For any set B $\subseteq$ X, $\psi\widehat{g}$-int(B) =($\psi\widehat{g}$ -cl$((B)^c)^c$)) . Thus, f($\psi\widehat{g}$-int(B)) = f(($\psi\widehat{g}$ -cl$((B)^c))^c$) = (f($\psi\widehat{g}$ -cl$(B^c)))^c$ = ($\psi\widehat{g}$ - cl$(B^c)))^c$, by theorem 3.6 ($\psi\widehat{g}$ -cl$(f(B^c)))^c$ = $\psi\widehat{g}$ -int(f(B)). ☐

Theorem 3.9. *If f:$(X,\tau) \rightarrow (Y,\sigma)$ is a $\psi\widehat{g}^*$-homeomorphism, then f^{-1} ($\psi\widehat{g}$-int(B))= $\psi\widehat{g}$-int(f^{-1}(B)) for all $B \subseteq Y$.*

Proof. Let f, g ϵ $\psi\widehat{g}^*$-h(X,τ).Then $g \circ f$ ϵ $\psi\widehat{g}^*$ -h(X,τ) and so $\psi\widehat{g}^*$-h(X,τ) is closed under the composition of functions. Composition of functions is always associative. The identity map I:$(X,\tau) \rightarrow (X,\tau)$ is a $\psi\widehat{g}^*$ -homeomorphisms and so Iϵ $\psi\widehat{g}^*$-h(X,τ). Also f $\circ$ I = I $\circ$ f = f for every f ϵ $\psi\widehat{g}^*$-h(X,τ) If f ϵ $\psi\widehat{g}^*$-h(X,τ) then f^{-1} ϵ $\psi\widehat{g}^*$-h(X,τ) and f $\circ$ f^{-1} =f^{-1} $\circ$ f = I. Hence $\psi\widehat{g}^*$-h(X,τ) is a group under the composition of functions. □

Theorem 3.10. *Let f:$(X,\tau) \rightarrow (Y,\sigma)$ be a $\psi\widehat{g}^*$-homeomorphism. Then f induces an isomorphism from the group $\psi\widehat{g}^* - h(X,\tau)$- onto the group $\psi\widehat{g}^*$-h (Y,σ).*

Proof. Let θ_f: $\psi\widehat{g}^*$-h$(X,\tau) \rightarrow \psi\widehat{g}^*$-h$(Y,\sigma)$ by defined as $\theta_f(h)$ = f $\circ$ h $\circ$ f^{-1} for every h ϵ $\psi\widehat{g}^*$-h(X,τ).Then θ_f is a bijection. Further, for all h_1, h_2 ϵ $\psi\widehat{g}^*$ - h(X,τ), $\theta_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h^{-1} \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta_f(h_1) \circ \theta_f(h_2)$. Therefore, θ_f is a homeomorphism and so it is an isomorphism induced by f. □

References

[1] Abd El-Monsef.M.E,Rose Mary.S and Lellis Thivagar.M.,*$\alpha\widehat{g}$-closed sets in topological spaces*, Assiut Univ.J. of Mathematics and Computer Science, No.1(36)(2007), 43-51.

[2] Andrijevic.D, *Semi-preopen sets*, Mat. Vesnik, 38(1)(1986), 24-32.

[3] Balachandran K, Sundaram P, Maki H , *On Generalized ho-meomorphisms in Topological Spaces*, Fukuoka University, Ed., part III.40, 13-21.

[4] Bhattacharya.P and B.K Lahiri,*Semi-generalized closed sets in topology*, Indian J.Math, Indian J.Math, 29(3)(1987), 375-382.

[5] Dontchev .J and M Ganster, *Onδ-generalized closed sets and T -spaces,*,Mem.Fac.Sci.Kochi Univ.Ser.A, Math., 17(1996), 15-31.

[6] Levine.N, *Generalized closed sets in topology* , Rend.Circ.Mat.Palermo, 19(1970) 89-96.

[7] Maki H, K. Balachandran, R.Devi, *Semi-generalized home-omorphisms and generalized semi homeomorphisms in Topological Spaces*, Indian J.of Closed Maps, J.Karnatk Univ. Sci. 27, (1982) 82-88.

[8] Najated.O,*On some classes of nearly open sets*, Pacific .J.Math., 15(1965) 961-970.

[9] Ramya.N and Parvathi.A, *$\psi\widehat{g}$-closed sets in topological spaces*, International Journal of Mathematical Archive, 2(10)(2011), 1992-1996.

[10] Ramya.N and Parvathi.A, *Strong forms of $\psi\widehat{g}$-continuous functions in topological spaces*, Journal of mathematics and computational sciences, 2 (2012), No. 1, 101-109.

[11] Ramya.N and Parvathi.A, *$(1,2)^* - \psi\widehat{g}$-closed functions in bitopological spaces*, International Journal of Mathematical Archive, 3(8)(2012), 3122-3128.

[12] Ramya.N and Parvathi.A *Quasi $\psi\widehat{g}$-open functions and Quasi $\psi\widehat{g}$ -closed functions in topological spaces*, Journal of Advanced Research in Computer Engineering, Volume 6, Number 2, July-December 2012, 103-106, ISSN 0974-4320.

[13] Ramya.N, *$\psi\widehat{g}$ -closed sets in BiCech closure spaces*, Asia Mathematika, Volume 2 Issue 1 (2018) page: 31-39.

[14] Veerakumar, M.K.R.S., *$\widehat{g}$-closed sets in topological spaces*,Bull. Allah.Math.Soc, 18(2003), 99-112.