Submaximality under mI\(\alpha\)g-closed sets in ideal minimal spaces

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Abstract: This article dealt a new submaximal space called mI\(\alpha\)g-submaximal space in ideal minimal space. Significant properties of mI\(\alpha\)g-submaximal space are studied. Equivalent conditions concerned with mI\(\alpha\)g-submaximal space and mI\(\alpha\)g-locally \(m^*\)-closed sets, \(m^*\)-codense sets, pre-m-I-open sets are also established

Key words: mI\(\alpha\)g-closed sets, mI\(\alpha\)g-locally \(m^*\)-closed sets, mI\(\alpha\)g-submaximal spaces

1. Introduction

Submaximality in general topological spaces was introduced by Hewit [7]. He defined a general way of constructing maximal topologies. A systemaized approach on submaximality in topological spaces is followed by Arhangel’skii et.al [1]. Necessary and sufficient conditions of submaximality are also proved by them. They have also proved that the submaximal space is left separated in a topological space. The concept, ideal in topological spaces was studied by Kuratowski [9] and Vaidyanathaswamy [16]. Several properties of ideal topological spaces were discussed by Jankovic et.al [8]. Properties of I-submaximal spaces was studied by Erdal Ekici et.al [6]. Ig-submaximal spaces was introduced by Bhavani et.al [3]. The concept of minimal structure and minimal spaces were introduced and studied by maki et.al [10]. Quite recently [11] Ozbakir et.al studied the impact of ideals in minimal spaces and termed as ideal minimal spaces. They have initiated \(A^*\)\(_m\), called the minimal local function in ideal minimal spaces. \(\alpha\)-generalised closed sets (briefly mI\(\alpha\)g-closed sets) in ideal minimal spaces was introduced and some significant properties were studied by Parimala et.al [2]. The concept Locally closed, on mI\(\alpha\)g-closed sets (briefly mI\(\alpha\)g-locally \(m^*\)-closed sets) was defined by Parimala et.al [12] In this paper we have proved some equivalent conditions of mI\(\alpha\)g locally \(m^*\)-closed sets. Further we have defined mI\(\alpha\)g-submaximal spaces in ideal minimal spaces and discussed significant properties of and mI\(\alpha\)g-submaximal spaces. Also we have proved several equivalent conditions on mI\(\alpha\)g-submaximal spaces.

2. Preliminaries

Definition 2.1. [9] Let \(X\) be a set and is non empty. Let \(I\) be the collection of subsets of \(X\) which is also non empty. \(I\) is referred as an ideal if it satisfies the conditions. Let \(A,B\) be any two subsets of \(I\) such that

(i) \(A \in I\) and \(B \subset A\) implies \(B \in I\)
(ii) \(A \in I\) and \(B \in I\) implies \(A \cup B \in I\).

Definition 2.2. [10] Consider a set \(X\) and \(M\) represents the set of all possible subsets of \(X\). \(M\) is termed as the minimal structure if \(\phi\) and \(X\) should be the members of \(M\). The minimal spaces we mean the set \(X\) with the minimal structure \(M\) say \((X,M)\). The elements of \(M\) are referred as \(m\)-open sets and their complements

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are called \(m\)-closed sets. The interior and closure of \(m\)-open sets are denoted by \(m\-\text{int}\) and \(m\-\text{cl}\) respectively and are defined as \(m\-\text{int}(A) = \bigcup \{ U : U \subset A, U \in M \}\), \(m\-\text{cl}(A) = \cap \{ F : A \subset F, X - F \in M \}\).

**Remark 2.3.** [11] The minimal space \((X, M)\) is said to exhibit the property \([U]\) if the union of any number of \(m\)-open sets is a \(m\)-open set and the property \([I]\) if the intersection of finite number of \(m\)-open sets is a \(m\)-open set.

**Definition 2.4.** [11] Let \(P(X)\) denotes the power set of \((X, M, I)\). The mapping \((\_)_m^* : P(X) \rightarrow P(X)\) leads the definition of the minimal local function \(A_m^*\) as \(\{ x \in X : U_m \cap A \notin I \}\) for all \(U_m \in U_m(x)\).

**Theorem 2.5.** [11] In a minimal space \((X, M)\), let \(I_1, I_2\) be two ideals on \(X\). \(K_1\) and \(K_2\) be subsets of \(X\). Then

\[
\begin{align*}
(a) & \quad K_1 \subset K_2 \Rightarrow K_1m^* \subset K_2m^*. \\
(b) & \quad K_1m^* \cup K_2m^* \subset (K_1 \cup K_2)_m^*. \\
(c) & \quad (K_m^*)_m \subset K_m^*. \\
(d) & \quad K_m^* = m\-\text{cl}(K_m^*) \subset m\-\text{cl}(K). \\
(e) & \quad I_1 \subset I_2 \Rightarrow K_m^*(I_2) \subset K_m^*(I_1).
\end{align*}
\]

**Remark 2.6.** [11] If the ideal minimal space \((X, M)\) includes the property \([I]\), then (b) of Theorem 2.5 satisfies the equality. That is \(K_1m^* \cup K_2m^* = (K_1 \cup K_2)_m^*\).

**Definition 2.7.** [11] The minimal \(*\)-closure operator \(m\-\text{cl}^*\) on a subset \(A\) of \((X, M, I)\) is defined as the union of \(A\) and \(A_m^*\). That is \(m\-\text{cl}^*(A) = A \cup A_m^*\). The minimal structure on \(m\-\text{cl}^*\) is termed as \(M^* (I, M)\) which is defined as \(M^* (I, M) = \{ F \subset X : m\-\text{cl}^*(X - F) = X - F \}\). The members of \(M^*\) are named as \(m^*\)-open sets. The interior of \(m^*\)-open sets is denoted by \(m\-\text{int}^*(A)\).

**Proposition 2.8.** [11] Salient features of the minimal \(*\)-closure operator \(m\-\text{cl}^*\) are listed below. Let \(K, K_1, K_2 \subset X\)

\[
\begin{align*}
(a) & \quad m\-\text{cl}^*(K_1) \cup m\-\text{cl}^*(K_2) \subset m\-\text{cl}^*(K_1 \cup K_2). \\
(b) & \quad If \ K_1 \subset K_2, \ then \ m\-\text{cl}^*(K_1) \subset m\-\text{cl}^*(K_2). \\
(c) & \quad K \subset m\-\text{cl}^*(K). \\
(d) & \quad m\-\text{cl}^*(\phi) = \phi \ and \ m\-\text{cl}^*(X) = X.
\end{align*}
\]

**Remark 2.9.** [11] When the ideal minimal space \((X, M, I)\) includes the property \([I]\) then equality holds in (a) of Theorem 2.8. That is \(m\-\text{cl}^*(K_1 \cup K_2) = m\-\text{cl}^*(K_1) \cup m\-\text{cl}^*(K_2)\) and also \(m\-\text{cl}^*(m\-\text{cl}^*(K)) = m\-\text{cl}^*(K)\) for the subset \(K\).

**Definition 2.10.** [11] In an ideal minimal space \((X, M, I)\), a subset \(A\) is termed as \(m^*\)-dense in itself set if \(A\) is a subset of \(A_m^*\) (briefly \(A \subset A_m^*\)).

**Lemma 2.11.** [11] If a subset \(A\) is seems to be a \(m^*\) dense in itself set in an ideal minimal space, then the following equality holds. \(A_m^* = m\-\text{cl}(A_m^*) = m\-\text{cl}(A) = m\-\text{cl}^*(A)\).

**Definition 2.12** [4] A subset \(A\) of \(X\) is defined as a \(m^*\)-dense set if \(m\-\text{cl}^*(A) = X\).
Theorem 3.3. Let $A$ be a non-empty subset of $X$, then $A$ is defined to be a mI-open-closed set if $A^*_m \subseteq U$ whenever $A \subseteq U$, $U$ is a $\alpha m$-open set.

Lemma 2.16. [4] A set $A$ is a $m^r$-closed set, then it is be a mI-open-closed set.

Definition 2.15. Consider a subset $A$ of $X$, then $A$ is defined to be a

1. I-locally $m^r$-closed set [14] if $A$ equals the intersection of a $m$-open set $U$ and a $m^r$-closed set $F$. That is $A = U \cap F$.

2. mI-open-closed set [12] if there exists a mI-open set $U$ and a $m^r$-closed set $F$ such that $A = U \cap F$.

3. mI-open-closed set [13] if $A \subseteq m$-int($m$-cl$(A)$).

Lemma 2.16. [4] $m^r$-dense sets are mI-open-closed sets.

Definition 2.17. [4] A subset $A$ of a minimal space $X$ is termed to be

(a) $m$-I-dense if $A^*_m = X$.

(b) $m^r$-codense set if $(X - A)$ is a $m^r$-dense set.

Definition 2.18. The ideal minimal space $(X, M, I)$ is defined to be a

(a) $m$-$T_1$-space [15] if there exists two elements $x, y$ of $X$ such that $x \cap y = \phi$, there exists a $m$-open set $P$ that contains $x$, but not $y$ and another $m$-open set $Q$ that contains $y$, but not $x$.

(b) $m$-$I$-submaximal space [5] if every $m^r$-dense subset of $X$ is a $m$-open set.

3. mI-open-closed set

Some characterisations of mI-open-closed $m^r$-closed set are as follows.

Theorem 3.1. The necessary and sufficient conditions of a subset $A$ to be a mI-open-closed set is $m$-cl$(A) \subseteq U$, $A \subseteq U$ where $U$ is a $m$-open set.

Proof. Necessity. Consider a mI-open-closed set $A$ and a $m$-open set $U$. Since $m$-cl$(A) = A \cup A^*_m$. Also we have $A \subseteq U$, $A^*_m \subset U$ and $U$ is a $m$-open set. Therefore $m$-cl$(A) \subseteq U$.

Proof. Sufficiency. Let $m$-cl$(A) \subseteq U$, $A \subseteq U$ and $U$ is a $m$-open set. Since $m$-cl$(A) = A \cup A^*_m \subseteq U$, we get $A, A^*_m \subseteq U$, $U$ is $m$-open. Since every $m$-open set is an $\alpha m$-opens set, $A^*_m \subseteq U$, $A \subseteq U$, $U$ is an $\alpha m$-open set. Therefore, $A$ is a mI-open-closed set.

Theorem 3.2. A $m$-closed set is always a mI-open closed set, equivalently $(A$ $m$-open set is a mI-open-closed set.)

Proof. Consider a $\alpha m$ open set $U$ and let $A \subseteq U$ be a $m$-closed set in $X$, then $m$-cl$(A) = A$, which implies $m$-cl$(A) \subseteq U$. It is clear that $m$-cl$(A) \subseteq m$-cl$(A) \subseteq U$. Since $m$-cl$(A) = A \cup A^*_m$ and $A \subseteq U$ we get $A^*_m \subseteq U$, where $U$ is a $\alpha m$-open set. Hence $A$ is a mI-open-closed set.

Theorem 3.3. If a subset $A$ of $(X, M, I)$ is an I-locally $m^r$-closed set, then it is a mI-open-closed $m^r$-closed set and the converse of this theorem may not be true explained in Example 3.4.

Proof. Referring Theorem 3.2, the proof follows from the definition of I-locally $m^r$-closed set.

Example 3.4. $(X, M, I)$ be an ideal minimal space with $X = \{a, b, c\}$, $M = \{\phi, X, \{a\}, \{b, c\}\}$ and $I = \{\phi, \{c\}\}$. In this example mI-open-closed $m^r$-closed sets are the elements of the power set of $X$, but the $\{b\}$ is not a locally $m^r$-closed set.
Theorem 3.5. Consider the ideal minimal space \((X, M, I)\) and \(A \subseteq X\) is \(m^*\)-dense in itself then statements below given are equivalent.

(a) \(A\) is an \(mI\alpha\)-locally \(m^*\)-closed set.

(b) \(A\) equals the intersection of a \(mI\alpha\)-open set \(U\) and \(m-cl^*(A)\). That is \(A = U \cap m-cl^*(A)\) for a \(mI\alpha\)-open set \(U\).

(c) \(m-cl^*(A) - A\) and \(A_m^* - A\) are equal and also \(mI\alpha\) closed sets.

(d) \(A \cup (X - m-cl^*(A))\) and \(A \cup (X - A_m^*)\) are equal and also \(mI\alpha\)-open sets.

(e) \(A \subset m-int(A \cup (X - A_m^*))\)

Proof. (a) \(\Rightarrow\) (b). If \(A\) is an \(mI\alpha\)-locally \(m^*\)-closed set, then \(A = U \cap F\) where \(U\) is \(mI\alpha\)-open and \(F\) is \(m^*\)-closed, that is \(F_m^* \subseteq F\). Since \(A = U \cap F\), we have \(A \subseteq U\) and \(A \subseteq m-cl^*(A)\). Therefore \(A \subseteq U \cap m-cl^*(A)\). Since \(F\) is \(m^*\)-closed and \(A \subseteq F\), \(m-cl^*(A) \subseteq m-cl^*(F)\), which implies \(U \cap m-cl^*(A) \subseteq U \cap m-cl^*(F)\). Then \(A = U \cap m-cl^*(A)\).

(b) \(\Rightarrow\) (c). Consider \(m-cl^*(A) - A = A_m^* - A = A_m^* \cap (X - A) = A_m^* \cap (X - (U \cap m-cl^*(A))) = A_m^* \cap (X - U)\). Let \(V\) be a \(m\)-open set such that \(m-cl^*(A) - A \subseteq V\). Then \(A_m^* \cap (X - U) \subseteq V\), which implies \((X - U) \subseteq ((X - A_m^*) \cup V)\). Since \(U\) is \(mI\alpha\)-open set \((X - U)\) is a \(mI\alpha\)-closed set. Therefore by Theorem 3.1, we get \(m-cl^*(X - U) \subseteq ((X - A_m^*) \cup U)\). Also \(A_m^* \cap m-cl^*(X - U) \subseteq V\). Consider \(A_m^* \cap (X - U) \subseteq A_m^*\) and \(A_m^* \cap (X - U) \subseteq (X - U)\). By Theorem 2.5(a) we get \((A_m^* \cap (X - U))_m \subseteq A_m^*\).

(d) \(\Rightarrow\) (e) Since \(A \subseteq (A \cup (X - A_m^*))\), \(m-int A \subset m-int(A \cup (X - A_m^*))\). Therefore \(A \subset m-int(A) \subset (A \cup (X - A_m^*))\) and hence \(A \subset m-int(A \cup (X - A_m^*))\).

(e) \(\Rightarrow\) (a) By (e) we can say \(A \cup (X - A_m^*) \subset m-int(A \cup (X - A_m^*))\). By (d) \(A \cup (X - m-cl^*(A)) = A \cup (X - A_m^*)\) is a \(mI\alpha\)-open set. Also \(A \cup (X - m-cl^*(A)) \cap m-cl^*(A) = (A \cap m-cl^*(A)) \cup ((X - m-cl^*(A)) \cap m-cl^*(A))) = (A \cap (A \cup A_m^*)) \cup \phi = A \cap (A \cap A_m^*) \subseteq A\) since \(A\) is a \(m^*\) dense set. Also since \(A\) is a \(m^*\) dense set, \(A \subseteq A_m^*\). Therefore \(m-cl^*(A) = A_m^*\). Hence \((m-cl^*(A))_m = (A_m^*)_m = A_m^* = m-cl^*(A)\). That is \((m-cl^*(A))_m \subseteq m-cl^*(A)\). Therefore \(m-cl^*(A)\) is a \(m^*\)-closed set. That is \(A = (A \cup (X - A_m^*) \cap m-cl^*(A))\) for a \(mI\alpha\)-open set and \(m-cl^*(A)\) is a \(m^*\)-closed set. Therefore \(A\) is \(mI\alpha\)-locally \(m^*\)-closed set.

Theorem 3.6. If \(G\) is a \(m\)-open subset of the ideal minimal space \((X, M, I)\), then \(G\) is \(mI\alpha\)-locally \(m^*\)-closed set, but the converse need not be true.

Proof. Lemma 2.14 and Theorem 3.2 proves the Theorem.

Example 3.7. Let the ideal minimal space with \(X = \{a, b, c\}, M = \{\emptyset, X, \{a\}, \{b, c\}\}\) and \(I = \{\emptyset, \{c\}\}\). In this example \(mI\alpha\)-locally \(m^*\)-closed sets are the elements of the power set of \(X\), but the \(\{b\}\) is not a \(m\)-open set.

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Theorem 3.8. A $mI$-locally $m^*$-closed set which is also a $m-I$-dense set, is a $mI$-open set.

Proof. Referring Theorem 3.5(d) if a set $A$ is a $mI$-locally $m^*$-closed set then $A \cup (X - m-cl^*(A))$ is a $mI$-open set. By the definition of $m-I$-dense set we get $A_m^* = X$. Therefore $m-cl^*(A) = A \cup A_m^* = X$, which implies $A \cup (X - X) = A$ is a $mI$-open set.

Corollary 3.9. Let $A$ be an $m-I$-dense subset of $X$, then $A$ is a $mI$-locally $m^*$-closed set if and only of $A$ is a $mI$-open set.

Theorem 3.10. Let $(X, M, I)$ be an ideal minimal space satisfying property $[I]$, then we can prove the equivalent statements on $mI$-locally $m^*$-closed sets.

(a) Every subset of $X$ is a $mI$-locally $m^*$-closed set.

(b) Every $m^*$-dense set a $mI$-open set.

Proof. (a) $\implies$ (b) The proof follows from Theorem 3.5(d).

(b) $\implies$ (a) Consider a subset $K$ of $X$. Let $S = K \cup (X - m-cl^*(K))$. Then $m-cl^*(S) = m-cl^*(K \cup (X - m-cl^*(K))) = (K \cup (X - m-cl^*(K)) \cup (K \cup (X - m-cl^*(K))))_m = (K \cup (X - m-cl^*(K))) \cup K_m^* \cup (X - m-cl^*(K)) = m-cl^*(K) \cup (X - m-cl^*(K)) = X$. That is we have proved $m-cl^*(S) = X$. Therefore by Definition 2.12 $S$ is a $m^*$-dense set. By (b) $S$ is a $mI$-open set. Therefore by Theorem 3.5(d), $K$ is a $mI$-locally $m^*$-closed set. 

Theorem 3.11. Let $(X, M, I)$ be an ideal minimal space satisfying the property $[U]$ and $A \subset X$, is a $m^*$-dense set. The necessary and sufficient condition that the set $A$ is a $mI$-locally $m^*$-closed set is $A \cap (A_m^* - A)_m^*$ is a $mI$-open set.

Proof. Let $A \subset X$. Consider $(m-cl^*(A) \cap (X - A))_m^* - (m-cl^*(A) \cap (X - A)) = (m-cl^*(A) \cap (X - A))_m^* \cap (X - m-cl^*(A) \cap (X - A)) = m-cl^*(A) \cap (X - A)_m^* \cap (X - m-cl^*(A)) \cup (m-cl^*(A) \cap (X - A)_m^* \cap A = (A_m^* - A)_m^* \cap A$ by Theorem 3.5(c) $A$ is a $mI$-locally $m^*$-closed set implies $m-cl^*(A) - A = m-cl^*(A) \cap (X - A)$ is a $mI$-open set. Hence $(m-cl^*(A) \cap (X - A))_m^* - (m-cl^*(A) \cap (X - A))$ is a $mI$-open set. That is $(A_m^* - A)_m^* \cap A$ is a $mI$-open set.

4. $mI$-Submaximal spaces

Definition 4.1. An ideal minimal space $(X, M, I)$ is said to be a $mI$-submaximal space if every $m^*$-dense subset in $X$ is a $mI$-open set.

Theorem 4.2. Every $mI$-submaximal space is a $mI$-submaximal space, but the converse of this statement may not be true.

Proof. Consider a $m^*$-dense set $A$ of a $mI$-submaximal space $X$. By the definition of $mI$-submaximal space $A$ is a $m$-open set. By Theorem 3.2, $A$ is a $mI$-open set in $X$. Therefore $(X, M, I)$ is a $mI$-submaximal space.

Example 4.3. Consider the ideal minimal space $(X, M, I)$ with $X = \{a, b, c, d\}$ and $M = \{\emptyset, X, \{a, b\}, \{b, c\}, \{a, c, d\}\}$ and the ideal $I = \{\emptyset, \{c\}\}$. $m^*$-dense sets are $\{\{a, b\}, \{a, b, c\}, \{a, b, c\}\}$. In this example $(X, M, I)$ is a $mI$-submaximal space, but not a $mI$-submaximal space since $\{a, b, d\}$ is a $m^*$-dense set, but not an $m$-open set.

Lemma 4.4. When $(X, M, I)$ represents a ideal minimal space and $A \subset M$, then below said statements are equivalent.

(a) $A$ is a $mI$-closed set.
(b) For a m-open set $U$ of $X$, $m-cl^*(A) \subset U, A \subset U$.

(c) Let $x \in m-cl^*(A)$, $m-cl(x) \cap A$ is non empty.

(d) $m-cl^*(A) - A$ does not have non empty m-closed sets.

(e) $A_m^* - A$ does not have non empty m-closed sets.

Theorem 4.5. If $(X, M, I)$ is a $m-T_1$ ideal space and $A$ is a $m^*$-dense in itself set and $mI\alpha g$-closed subset of $X$, then $A$ is a $m$-closed set.

Proof. Let $A$ be $m^*$-dense in itself set and $mI\alpha g$-closed subset of $X$. Then by Lemma 4.4 there exists no non empty $m$-closed set in $m-cl^*(A) - A$. Also since $X$ is a $m-T_1$, $A$ is a $m^*$-closed set we have $m-cl^*(A) - A = \phi$. $A$ is a $m^*$-closed set implies $m-cl^*(A) = m-cl(A)$. Therefore $m-cl(A) - A = \phi$ which implies $m-cl(A) = A$. Hence $A$ is a $m$-closed set.

Lemma 4.6 In a ideal minimal space $(X, M, I)$ if a subset $A$ is a pre-m-I-open set, then $A$ can be expressed as $A = G \cap B$, where $G$ is a $m$-open set and $B$ is a $m^*$-dense set.

Proof. Consider a pre-m-I-open set $A$, we have $A \subseteq m-int(m-cl^*(A)) = U$ where $U$ is a $m$-open set. Let $D = X - (U - A) = X - (U \cap A^c) = X \cap (U \cap A^c)^c = X \cap (U^c \cup A) = (U^c \cup A) = (X - U) \cup A$. To prove $D$ is a $m^*$-dense set it is enough to prove $m-cl^*(D) = X$. That is $m-cl^*((X - U) \cup A) \supseteq m-cl^*(X - U) \cup m-cl^*(A) \supseteq (X - m-cl^*(A) \cup m-cl^*(A)) = X$. Therefore $D$ is $m^*$-dense set. Also $U \cap D = U \cap ((X - U) \cup A) = (U \cap (X - U)) \cup (U \cap A) = \phi \cup (U \cap A) = \phi \cup A = A$ here $A \subseteq U$. That is $A = U \cap D$ such that $U$ is a $m$-open set and $D$ is a $m^*$-dense set.

Remark 4.7. Property [j] refers ”union of two $mI\alpha g$-closed sets is a $mI\alpha g$-closed set” and property [v] refers "For any two $mI\alpha g$-closed sets $A$ and $B$, $A \cap B$ is also a $mI\alpha g$-closed set”.

Theorem 4.8. Let $(X, M, I)$ be the ideal minimal space satisfying property [v] then statements below which are equivalent.

(a) $(X, M, I)$ is a $mI\alpha g$-submaximal space.

(b) If $A$ is a pre-m-I-open set, then $A$ is a $mI\alpha g$-open set.

Proof. (a) $\implies$ (b). Let $(X, M, I)$ is a $mI\alpha g$-submaximal space and $A \subset X$ be a pre-m-I-open set. By Lemma 4.6 $A = U \cap D$, $U$ is a $m$-open set and $D$ is a $m^*$-dense set. Since $X$ is a $mI\alpha g$-submaximal space and $D$ is a $mI\alpha g$-open set in $X$ and $U$ is a $m$-open set, implies $U$ is a $mI\alpha g$-open by Theorem 3.2, therefore $A$ is a $mI\alpha g$-open set.

(b) $\implies$ (a). Let $A$ be $m^*$-dense subset of $X$, then by Lemma 2.16 $A$ is a pre-m-I-open set. By hypothesis $A$ is a $mI\alpha g$-open set. Therefore $X$ is a $mI\alpha g$-submaximal space.

Theorem 4.9. Let $(X, M, I)$ be the ideal minimal space satisfying property [v] then the following statements are equivalent.

(a) $(X, M, I)$ is a $mI\alpha g$-submaximal space.

(b) If $A$ is a subset of $X$, then $A$ is a $mI\alpha g$-locally $m^*$-closed set.

(c) Any $m^*$-dense subset of $X$ is the intersection of a $m^*$-closed set and a $mI\alpha g$-open subset of $X$. 

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Proof. \((a) \implies (b)\). Let \((X, M, I)\) be a \(m\Omega g\)-submaximal space, by definition of \(m\Omega g\)-submaximal space every \(m^*\)-dense set is a \(m\Omega g\)-open set. Referring Theorem 3.10, every subset is a \(m\Omega g\)-locally \(m^*\)-closed set.

\((b) \implies (c)\). Let \(A\) be a \(m^*\)-dense set, by \((a)\) of this theorem \(A\) is \(m\Omega g\)-locally \(m^*\)-closed set. Theorem 3.5(b) explains, there exists a \(m\Omega g\)-open set \(U\) such that \(A = U \cap m\text{-cl}^*(A)\). Consideration of \(A\) as a \(m^*\)-dense set we inferred that \(m\text{-cl}^*(A) = X\). Hence \(A = U \cap m\text{-cl}^*(A) = U \cap X = U\) and \(U\) is a \(m\Omega g\)-open set in \(X\).

\((c) \implies (a)\). Let \(A\) be \(m^*\)-dense set in \((X, M, I)\), by \((c)\) \(A = U \cap E, U\) is a \(m\Omega g\)-open set and \(E\) is a \(m^*\)-closed set. Since \(A \subseteq E\), \(E\) is \(m^*\)-dense set. That is \(m\text{-cl}^*(E) = X\). Since \(E\) is a \(m^*\)-closed set \(m\text{-cl}^*(E) = E = X\). Hence \(A = U \cap E = U \cap X = U\) and it is a \(m\Omega g\)-open set. Therefore \((X, M, I)\) is \(m\Omega g\)-submaximal space.

Theorem 4.10. For an ideal minimal space, the following statements are equivalent.

\[(a)\] \((X, M, I)\) is a \(m\Omega g\)-submaximal space.

\[(b)\] Whenever the subset \(A\) is not a \(m\Omega g\)-open set, we can prove \(A - (m\text{-int}(m\text{-cl}^*(A))) \neq \emptyset\).

Proof. \((a) \implies (b)\). Assume the contrary that \(A - (m\text{-int}(m\text{-cl}^*(A))) = \emptyset\). Therefore \(A \subseteq (m\text{-int}(m\text{-cl}^*(A)))\), means \(A\) is a pre-\(m\Omega\)-open set. Since \(X\) is a \(m\Omega g\)-submaximal space, by Theorem 4.8, \(A\) is \(m\Omega g\)-open. Which is a contradiction to our assumption that \(A\) is not a \(m\Omega g\)-open set. Therefore \(A - (m\text{-int}(m\text{-cl}^*(A)))\) is non empty.

\((b) \implies (a)\). Consider a pre-\(m\Omega\)-open set \(A\) which is not a \(m\Omega g\)-open set, then by \((b)\) \(A - (m\text{-int}(m\text{-cl}^*(A)))\) is non empty, which implies \(A \nsubseteq (m\text{-int}(m\text{-cl}^*(A)))\). That is \(A\) may not be a pre-\(m\Omega\)-open set. Which contradicts to our assumption that \(A\) is Pre-\(m\Omega\)-open. Therefore \(A\) is a \(m\Omega g\)-open set and hence referring Theorem 4.8 \(X\) is \(m\Omega g\)-submaximal space.

Theorem 4.11. Consider an ideal minimal space \((X, M, I)\) with the property \([v]\), then it is possible for the equivalent statements on \(m\Omega g\)-submaximal spaces.

\[(a)\] \((X, M, I)\) is a \(m\Omega g\)-submaximal space.

\[(b)\] The family of all \(m\Omega g\)-open sets \(\zeta\) such that \(\zeta = \{U - A : U\) is \(m\Omega g\)-open and \(m\text{-int}^*(A) = \emptyset\}\).

Proof. \((a) \implies (b)\). Let \(X\) \(m\Omega g\)-submaximal space. Construct a \(\sigma\) as \(\sigma = \{U - A : U\) is \(m\Omega g\)-open and \(m\text{-int}^*(A) = \emptyset\}\). Our aim is to prove \(\sigma = \zeta\). Consider an element \(G \in \zeta\). Since \(G = G - \emptyset\) and \(m\text{-int}^*(\emptyset) = \emptyset\), \(G \in \sigma\).

Hence \(\zeta \subseteq \sigma\). \((1)\)

Let \(G \in \sigma\). It is sufficient to prove \(G\) is a \(m\Omega g\)-open set. Since \(G \in \sigma\), \(G\) can be written as \(G = U - A\) such that \(U\) is a \(m\Omega g\)-open set and \(m\text{-int}^*(A) = \emptyset\). Also \(G = U - A = U \cap (X - A)\). Since \(m\text{-int}^*(A) = \emptyset\), \(X - (m\text{-int}^*(X - A)) = m\text{-cl}^*(A) = X\). That is \((X - A)\) is a \(m^*\)-dense set. Since \(X\) is a \(m\Omega g\)-submaximal space, \((X - A)\) is a \(m^*\)-dense set implies \((X - A)\) is a \(m\Omega g\)-open set. Hence \(G = U \cap (X - A)\) is a \(m\Omega g\)-open set and so \(\sigma \subseteq \zeta\). \((2)\)

combining equations \((1)\) and \((2)\) we have proved \(\sigma = \zeta\). Hence \((b)\) can be followed.

\((b) \implies (a)\). Consider \(A\) be a pre-\(m\Omega\)-open set. With reference of Lemma 4.6. \(A\) can be written as the intersection of the sets \(G\) and \(B\) such that \(G\) is a \(m\)-open set and \(B\) is a \(m^*\)-dense set. Therefore \(m\text{-cl}^*(B) = X\) and so \(m\text{-int}^*(X - B) = \emptyset\). That is \(A = G \cap B = G - (X - B)\) and \(m\text{-int}^*(X - B) = \emptyset\). By Theorem 3.2 \(G\) is a \(m\Omega g\)-open set. Therefore \(A \subseteq \zeta\) and hence \(A\) is a \(m\Omega g\)-open set. That is assumption of
the pre-mI-open set $A$ leads that $A$ is mI$\alpha$-open. Hence Theorem 4.8. iners that $X$ is a mI$\alpha$-submaximal space.

**Theorem 4.12.** Let $(X, M, I)$ be an ideal minimal space satisfying the property $[I]$ then the equivalent statements on mI$\alpha$-submaximal spaces are as follows.

(a) $X$ is a mI$\alpha$-submaximal space.

(b) There exists a mI$\alpha$-closed set $m$-$cl^*(A) - A$ for every subset $A$ of $X$.

**Proof.** $(a) \implies (b)$. Consider a mI$\alpha$-submaximal space $(X, M, I)$ and let $A \subset X$. To prove $m$-$cl^*(A) - A$ is mI$\alpha$-closed set is sufficient to prove $X - (m$-$cl^*(A) - A)$ is mI$\alpha$-open. We need to prove $X - (m$-$cl^*(A) - A)$ is a $m^\ast$-dense set. It is necessary to prove that $m$-$cl^*(X - (m$-$cl^*(A) - A)) = X$. Proceeding as, $m$-$cl^*(A \cup (X - (m$-$cl^*(A)))) = m$-$cl^*(A) \cup m$-$cl^*(X - (m$-$cl^*(A))) = X$. Therefore $(X - (m$-$cl^*(A) - A))$ is a $m^\ast$-dense set. Also since $X$ is a mI$\alpha$-submaximal space, by the definition of mI$\alpha$-submaximal space every $m^\ast$-dense set is a mI$\alpha$-open set. Hence $(X - (m$-$cl^*(A) - A))$ is a mI$\alpha$-open set. Therefore $(m$-$cl^*(A) - A)$ is a mI$\alpha$-closed set.

(b) $(\implies a)$. Let $(m$-$cl^*(A) - A)$ be a mI$\alpha$-closed set and let $A \subset X$ be a $m^\ast$-dense set in $(X, M, I)$. Therefore $m$-$cl^*(A) = X$, implies $(X - A)$ is a mI$\alpha$-closed set. So $A$ is mI$\alpha$-open. $(X, M, I)$ is mI$\alpha$-submaximal space.

**Theorem 4.13.** Consider a $m^\ast$-dense subset $A$, which is also a $m^\ast$-dense in itself set in a ideal minimal space $(X, M, I)$, then the equivalent statements are as follows.

(a) $(X, M, I)$ is a mI$\alpha$-submaximal space.

(b) $A \cap (A^*_{m} - A)_m^*$ is a mI$\alpha$-open set for every $A \subset X$.

**Proof.** $(a) \implies (b)$. Let $(X, M, I)$ is a mI$\alpha$-submaximal space. By Theorem 4.9, every subset of $A$ of $X$ is a mI$\alpha$-locally $m^\ast$-closed set. Then by Theorem 3.11 $A \cap (A^*_{m} - A)_m^*$ is a mI$\alpha$-open set in $X$.

(b) $(\implies a)$. By hypothesis $A \cap (A^*_{m} - A)_m^*$ is mI$\alpha$-open for every $A \subset X$. Since $A$ is a $m^\ast$-dense set we have $m$-$cl^*(A) = X$. So that $A^*_{m} - A = m$-$cl^*(A) - A = X - A$. Hence $A \cap (A^*_{m} - A)_m^* = A \cap (X - A)_m^* = (X - A)_m^* - (X - A)$ is a mI$\alpha$-open set and hence $(X - A)$ is a mI$\alpha$-closed set, implies $A$ is a mI$\alpha$-open set. Therefore $(X, M, I)$ mI$\alpha$-submaximal space.

**Theorem 4.14.** For a topological space with minimal structure $M$ and the ideal $I$, then the following statements are equivalent.

(a) $(X, M, I)$ is a mI$\alpha$-submaximal space.

(b) If a subset $A$ is a $m^\ast$-codense subset of $X$, then it is a mI$\alpha$-closed set.

**Proof.** $(a) \implies (b)$. Let $A$ be a $m^\ast$-codense set, then $(X - A)$ is a $m^\ast$-dense set. By Theorem 4.13, $(X - A)$ is a mI$\alpha$-open set. Hence $A$ is a mI$\alpha$-closed set.

(b) $(\implies a)$. Since $A$ is a $m^\ast$-codense set $(X - A)$ is a $m^\ast$-dense set and so $(X - A)$ is a mI$\alpha$-open set. Therefore by Theorem 4.13 $(X, M, I)$ is a mI$\alpha$-submaximal space.

**Theorem 4.15.** Let $(X, M, I)$ be an ideal minimal space, where $I$ is a $m$-codense ideal. If every subset is m-I-locally -m-closed, then $(X, M, I)$ is a mI$\alpha$-submaximal space.

**Proof.** Let $A$ be $m^\ast$-dense. Since $I$ is $m$-codense, $A$ is ,m-I-dense and m-I-locally -m-closed, then $A$ is
m-open and hence $mI\alpha g$-open. Therefore $X$ is $mI\alpha g$-submaximal space.

**Theorem 4.16.** In a $mI\alpha g$-submaximal space $(X, M, I)$ if for any two ideals $I, I'$ such that $I \subset I'$, then $(X, M, I')$ is a $mI'\alpha g$-submaximal space.

**Proof.** Let $A$ be $m^*(I')$-dense subset in $(X, M, I')$, then $A \cup A^*_m(I') = X$. As $I \subset I'$ we get $A^*_m(I') \subset A^*_m(I)$, which implies that $X = A \cup A^*_m(I') \subset A \cup A^*_m(I)$. That is $A \cup A^*_m(I) = X$. Thus $A$ is a $m^*(I)$-dense set. As $(X, M, I)$ is $mI\alpha g$-submaximal, $A$ is $mI\alpha g$-open. Since $I \subset I'$, $A$ is $mI'\alpha g$-open. Therefore $(X, M, I')$ is $mI\alpha g$-submaximal space.

5. **Conclusion**

In this article we have discussed some salient features of $mI\alpha g$-locally $m^*$-closed sets. We have introduced a new submaximality called $mI\alpha g$-submaximality in ideal minimal spaces and studied its significant features. Equivalence of $mI\alpha g$-submaximality with Pre-$mI$-open sets, $mI\alpha g$-locally $m^*$-closed sets, $m^*$-codense sets are given. Heredity nature of ideals are imported under $mI\alpha g$-submaximality. Future work of this article will be in hyper connectedness and $mI\alpha g$-submaximality.

**References**


