Asia Mathematika
Volume: 3 Issue: 2 , (2019) Pages: 34 – 52
Available online at **www.asiamath.org**

On Certain Distance Graphs and related Applications

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Received: 16 Apr 2019 • **Accepted:** 26 May 2019 • **Published Online:** 01 Aug 2019

Abstract: Stimulated by the famous plane coloring problem Eggleton coined the term distance graph and studied widely the prime distance graphs. A prime distance graph (PDG) $G(Z, D)$ is one whose vertex set V is the set of integers Z and the distance set D is a subset of the set of primes P. The edge set of G denoted E is the one whose elements (u, v) for any $u, v \in V(G)$ are characterized by the property that $d(u, v) \in D$ where $d(u, v) = |u - v|$. According to J.D.Laison, C. Starr and A. Walker a graph G is a PDG if there exists a 1-1 labeling $f : V(G) \to Z$ such that for any two adjacent vertices u and v the integer $|f(u) - f(v)|$ is a prime. Further they called such a labelling of $V(G)$ a prime distance labelling (PDL) of G. In this paper we prove certain existence and non-existence results concerning PDG and PDL and study the relationship between them. We also discuss certain applications besides raising some open problems.

Key words: Prime Distance Graph; Prime Distance Labeling; Chromatic number; Pythagorean number; Pythagorean triples and Pythagorean quadruples.

1. Introduction

On Pythagorean Triples and Quadruples

A Pythagorean triple (x, y, z) of positive integers x, y and z can be thought of as the sides of a right angled triangle and hence $x^2+y^2 = z^2$. One can generate out of a given such triple (x, y, z), other similar triples of the form $r(x, y, z)$, where $r > 1$. A primitive pythagorean triple (PPT) is one whose elements are pairwise relatively prime. It is interesting to observe that pythagorean triples were first found on cuneiform tablets of Babylon [34]. They are pertinent in the design of the altars of vedic rituals and the method of using them were described in ancient geometry books of India [11, 26, 39, 43] and also seen in the findings of Euclid and Diophantus. A first and early mention of the pythagoras theorem also referred as the square on the diagonal, along with some illustrations can be seen in the geometry book of Baudhāyana (c. 800 BC). It is said in his Śulba Sūtra 1.12 and 1.13 of [39] that: The areas of the squares made separately by the length and the breadth of a rectangle together equal the area of the square made by the diagonal. This is noted in rectangles with sides 3 and 4, 12 and 5, 15 and 8, 7 and 24, 12 and 35, and 15 and 36. To know more about Pythagorean triples and quadruples and their indexing one can consult [2, 33]. According to O'Conner and Robertson the period of Baudhāyana is around 800 BC [35]. Further Seidenberg [38] provides a number of arguments for the familiarity of pythagorous theorem in India much before pythagorous. One can refer for other aspects of ancient Indian mathematics to [27, 28, 36] and for the altar ritual that provides the context in which this mathematics was used in the Śulba Sūtras in [24, 25]. Van der Waerden noticed a ritual origin to the discovery of Pythagoras [45]. Pythagorous theorem considered as a remarkable discovery by eminent scholars of ancient east

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and ancient Greece is believed as the beginning of the basis of modern mathematics and of theoretical physics. It gives rise to several unexpected problems such as irrational numbers and incommensurability segments. The study of pythagorean 3-tuples dates back to 1000 years ahead of the pythagorous about 585-147 BC. This fact is substantiated by markings made on Babylonian tablets about the 3-tuples (3,4,5) and (4961, 6480, 8161).

On Graphs and Graph Coloring

The graphs considered in this paper are all finite, simple and undirected.

In a variety of contexts, the most pertinent feature to look and explain is the connections between some objects. Suppose that a college student is about to register for her forthcoming semester's classes. She may collect a list of the classes being offered and how many of them have prerequisites. This relation between courses namely which course is a prerequisite for another course is one of the fundamental priority of her wish list that she endeavors to identify. Next if an automobile parts manufacturing agent wants to find cost efficient routes to transport materials and supplies between various buildings on the campus of its major plants. Establishing viable paths between buildings are essential to craft cost efficient routes. These and many other practical scenarios can be modeled with graphs.

One of the vital topics in graph theory research is graph coloring. Fascinating and worthwhile generalizations of the ideas of graph coloring stems from the problems concerning channel allotment in wireless communications, traffic regulating, fleet maintenance, task allotment, and other applications. One can refer [37] for more. In traditional vertex coloring problems [19] a constraint is imposed only on colors of adjacent vertices. But meaningful generalizations demand colors to accommodate stronger constraints such as colors both of adjacent vertices and of vertices at distance 2 in the graph.

A graph is a 2-tuple $G = (V, E)$ where V is the vertex set of the graph and E is the edge set of the graph. The edge set E consists of unordered pairs of elements from V. The elements of V are called vertices and elements of E are called edges. For basics and terminology one can refer to [4, 18]. Vertex coloring is a very common graph coloring problem. The problem asks for, given a set of m colors, find a procedure for coloring the vertices of a graph such that no two adjacent vertices are colored with the same color. The graph coloring problem has a variety of practical applications such as drafting a schedule or time-table, in mobile radio frequency allotment, in suduku, in register allocation, in bipartite graphs, map coloring, etc., For instance, Suppose that X runs a network of lakhs of servers to propagate content on Internet. X install a latest software or update existing ones every week. The update cannot be done simultaneously on every server as it has to be taken down for the install. Further the update cannot be done one at a time as it takes a lot of time. Moreover certain servers cannot be taken down together as they perform some critical functions. Easily it is a scheduling application of graph coloring problem. It is demonstrated that 8 colors are required to color a graph of 75000 vertices. This means X can realise install updates in 8 passes. For more such examples one can refer to [30] which is a case study on graph coloring applications.

Distance Graphs and their Coloring

Suppose that (X, ρ) is a metric space with ρ as distance function) and let D be a subset of R^+. Frequently D will be a singleton and in certain cases it is set as $D = \{1\}$. The distance graph $G(X, D)$ is one whose vertices are the elements of X with $(x, y) \in E(G)$ exactly when $\rho(x, y) \in D$. We say that the graph G has chromatic number $\chi(G) = k \in N$ if there is a k-coloring of G. That is, it is a map from $V(G)$ to $\{1, \ldots, k\}$ that assigns adjacent vertices to different values and if k is minimal with this property. We deem

that $\chi(G) \leq \aleph_0$ if there is a coloring with values in N, and $\chi(G) = \aleph_0$ if in addition there is no k-coloring for any finite k.

The graph $G(X, D)$ is mostly not locally finite, but in many interesting instances it has finite chromatic number. This fact is attributed to a theorem of De Bruijn-Erdös [12], that employs the axiom of choice to establish a fact that a graph is k-colorable if and only if all its finite subgraphs are k-colorable. Observe that the chromatic number of $G(X, \{1\})$ may depend on the chosen set of axioms in some cases [40]. The chromatic number of $G(X, D)$ shall be denoted by $\chi(G(X, D))$ or $\chi(X, D)$. A long-standing open question [13] is to determine the chromatic number $\chi(E^2, \{1\})$ where E^2 is the Euclidean plane R^2 with the usual distance metric. We know that $4 \leq \chi(E^2, \{1\}) \leq 7$ soon after the question was raised but no more beyond that even to date. Lately it was showed that Lebesgue-measurable colorings of $G(E^2, \{1\})$ must require at least 5 colors [17, 42]; more specifically, $\chi(E^2, \{1\}) \geq 5$ under replacement axioms for the axiom of choice including the axiom that all subsets of R are Lebesgue measurable. The higher dimensional Euclidean spaces have been well considered in [13], and their variations in [9]. But then a very little attention has been given to the case when X is not Euclidean. Simmons considered the round spheres in [41]; Chilakamarri [8] considered the Minkowski planes (that is., R^2 endowed with a norm) and showed interesting relationship with the Euclidean case; and [21] briefly considered the more general case when R^2 is endowed with a translation-invariant distance that induces the usual topology. Johnson and Szlam asked several questions, some of which Benoît R. Kloeckner answered in [3]. He moved further away from Euclideanness by studying the case when $X = H^2$, the hyperbolic plane. This was suggested by Matthew Kahle on MathOverflow [23]. As H^2 has no homothety, even when D is a singleton the choice of the value matters a lot. As a consequence, the question of determining the behavior of $\chi(H^2, \{d\})$ when d varies seems as rich as the question of the relation between $\chi(E^n, \{1\})$ and n.

Integer Distance Graphs

If $D \subseteq Z^+$, then the integer distance graph $G(Z, D)$ is the graph with vertex set $V(G(Z, D) = Z$ and two vertices x and y are adjacent iff $|x - y| \in D$. Integer distance graphs were studied first by Eggleton et. al in [14]. A large number of papers were written on this topic, see [7, 15, 29, 46–49]. The chromatic number of $G(Z, P) = 4$ when P is a set of primes was done in [14]. The chromatic number of $G(Z, D)$ is completely found when D contains at most three elements. Clearly $\chi(G(Z, D)) = 2$ if D is a one element set. $\chi(G(Z, D)) = 2$ if D contains only odd elements. This is because $|D| + 1$ is an easy upper bound for $\chi(G(Z, D))$ if D is finite [14, 49]. $\chi(G(Z, D)) = 3$ if the elements of D includes two coprime elements of distinct parity.

2. Non-existence of certain PDGS'

Fermat established the non existence of a pythagorean triangle with side lengths as integers and area, a square of an integer. Mohanty et.al in [44] coined the term pythagorean number for the area of a pythagorean triangle. We infer from the above that the credit for establishing the fact: A pythagorean number cannot be a square of an integer goes to Fermat.

Theorem 2.1. *Let $k \equiv 3 \pmod 8$ be an odd square free integer such that it is either (i) a prime p_i with $p_i \equiv 3 \pmod 8$ or (ii) a product of two primes p_1, p_2 with former congruent to 5 (mod 8) and the latter congruent to 7 (mod 8) and p_1 a quadratic non residue of p_2 or (iii) a product of n distinct primes $p_1, \ldots, p_n$, $n \geq 2$ with the first one congruent to 3 (mod 8) and all the others congruent to 1 (mod 8) and further for $2 \leq j \leq n$ each of p_j's are quadratic residues of each other and all of them quadratic non residues of p_1 or*

(iv) a product of n distinct primes p_j, $j = 1, \ldots, n$ with the first prime p_1 congruent to 5 (mod 8), the second prime p_2 congruent to 7 (mod 8) and all the others congruent to 1 (mod 8) and further with the property: p_1 a quadratic non residue of p_2 and for $3 \leq j \leq n$ p_j's a quadratic residue of each other and hence of p_1 and p_j's a quadratic non residue of p_2 or vice-versa. Then a pythagorean distance graph with distance set elements equaling k-times an integer square does not exist.

Proof. Suppose that L, M, N forms a pythagorean triple and the area of the corresponding pythagorean triangle is $1/2LM = ks^2$ (A). It is elementary to note that we can deem $(L, M) = 1$. This is because in the otherwise instance, the problem simplifies to the case of a pythagorean triple L_1, M_1, N_1 with $(L_1, M_1) = 1$ and $1/2L_1M_1 = ks_1^2$. So we let L, M, N to be a primitive pythagorean triple by taking $L = p^2 - q^2$, $M = 2pq$ and $N = p^2 + q^2$ for $p, q \in Z^+$ with $(p, q) = 1$ and $p + q$ congruent to 1 (mod 2). Then (A) implies $(p-q)(p+q)pq = ks^2$ (B). So $(p, q) = (p, p+q) = (p, p-q) = (q, p+q) = (q, p-q) = (p-q, p+q) = 1$ (C). Now as k is square free four cases arise:

Case 1: Only one of $p + q$, $p - q$, p or q is k times an integer square and the other three are integer squares.

Assume that $p = kp_1^2$, $q = q_1^2$, $p - q = q_2^2$, $p + q = q_3^2$. Then $kp_1^2 - q_1^2 = q_2^2$ and $kp_1^2 + q_1^2 = q_3^2$. So $2kp_1^2 = q_2^2 + q_2^3$ and $(q_2, q_3) = 1$ by (C). This is a contradiction as no $sp - q = q_2^2$, $p + q = q_3^2$. Then $p_1^2 - kq_1^2 = q_2^2$, $p_1^2 + kq_1^2 = q_3^2$. As $p + q \equiv 1 \pmod 2$ we deduce that $p_1 + q_1 \equiv 1 \pmod 2$. We note that with $k \equiv 3 \pmod 4$, $p_1 \equiv 1 \pmod 2$ and $q_1 \equiv 0 \pmod 2$. Further $(p_1, q_1) = 1$ implies that $(q_1, q_2) = 1$. So $2p_1^2 = q_2^2 + q_3^2$. Moreover as $p, q \neq 0$, p_1, q_2, q_3 are all positive and $(q_2, q_3) = 1$. So we deduce that $p_1 = l^2 + m^2$, $q_2 = l^2 + 2lm - m^2$, $q_3 = m^2 + 2lm - l^2$ for $l, m \in Z^+$ with $l + m \equiv 1 \pmod 2$ and $(l, m) = 1$. So $kq_1^2 = q_3^2 - p_1^2 = (q_2 - p_1)(q_2 + p_1) = (2lm - 2l^2)(2m^2 + 2lm) = 4lm(m-l)(m+l)$. That is $kq_4^2 = (m-l)(m+l)lm$ where $q_1 = 2q_4$. Hence $(m^2 - l^2,\ 2lm,\ l^2 + m^2)$ is a primitive Pythagorean triple whose area is kq_4^2. But $kq_1^2 = q_3^2 - p_1^2 \leq q_3^2 = p + q$. So $0 < m + l < p + q$. This implies that an infinite descent with respect to (B) is obtained. Now if $p = p_1^2$, $q = q_1^2$, $p - q = kq_2^2$, $p + q = q_3^2$ then $p_1^2 - q_1^2 = kq_2^2$, $p_1^2 + q_1^2 = q_3^2$. Therefore $2p_1^2 = kq_2^2 + q_3^2$. As $q_2 \equiv q_3 \equiv 1 \pmod 2$ we deduce that $2p_1^2 \equiv k + 1 \pmod 8$. So $k \equiv 2p_1^2 - 1 \equiv \pm 1 \pmod 8$. But $k \equiv 3 \pmod 8$, a contradiction. To conclude, if $p = p_1^2$, $q = q_1^2$, $p - q = q_2^2$, $p + q = kq_3^2$ then this also results in a contradiction as $p + q = p_1^2 + q_1^2 = kq_3^2$, $k \equiv 3 \pmod 4$ and $(p_1, q_1) = 1$.

Case 2: Out of $p + q$, $p - q$, p and q one of them is x times a square, another one is y times a square and the remaining two are squares with $xy \equiv k \equiv 3 \pmod 8$ with $1 < x, y < k$.

As $xy \equiv 3 \pmod 8$ implies either $x \equiv 3$, $y \equiv 1 \pmod 8$ or the other way else $x \equiv 5$, $y \equiv 7 \pmod 8$ or the other way. Without loss of generality assume that $x \equiv 1$, $y \equiv 3 \pmod 8$. $xy = k$ with $1 < x, y < k$ points to the fact that k belongs to category (iii) or (iv) of the statement of the theorem. If k belongs to the category (iii), then $k = \prod_{i=1}^{n} p_i$ with $p_i \equiv 3 \pmod 8$ and $p_2 \equiv p_3 \equiv \cdots \equiv p_n \equiv 1 \pmod 8$. Also, $x = \prod_{j=1}^{k} q_j$ and $y = p_1$ or $y = p_1 \prod_{j=k+1}^{n-1} q_j$ where the two sets of q's are with the property that their union is $\{p_2, p_3, \ldots, p_n\}$ and are disjoint. Now all the different possibilities of this case 2 takes us to the congruence of the form $y.Y^2 \equiv w.W^2 \pmod{q_1}$, with $(yY, q_1) = 1$ and where $w = 1, -1, 2$ or -2. So $q_1 \equiv 1 \pmod 8$ implies y is a

quadratic residue of q_1. But by our hypothesis, p_1 is a quadratic non residue and q_j's for $j = k+1, \ldots, n-1$ are all quadratic residues of q_1. This means, y is a quadratic non residue of q_1 a contradiction. If k belongs to the category (iv) of the theorem $k = \prod_{i=1}^{n} p_i$ with $p_1 = 5$, $p_2 = 7$ and $p_3 \equiv p_4 \equiv \ldots p_n \equiv 1 \pmod 8$. So as in the above argument, $x = \prod_{j=1}^{k} q_j$, and $y = p_1 p_2$ or $y = p_1 p_2 \prod_{j=k+1}^{n-2} q_j$, where the two sets of q_j's are such that their union is $\{p_3, \ldots, p_n\}$ and disjoint. This means y is a quadratic residue of q_1. But by the hypothesis each q_j for $j = k+1, \ldots, n-2$ are quadratic residues of q_1. So either p_1 is a quadratic residue of q_1 and p_2 is a quadratic non residue of q_1 or p_1 is a quadratic non residue of q_1 and p_2 is a quadratic residue of q_1. In any of these occurrence we infer that y must be a quadratic non residue of q_1. This contradiction ensures the completion of discussion of this sub-occurrence. As the same logic extends to all other sub occurrences and cases we omit them and conclude that a Pythagorean distance graph with distance set elements as k times an integer square does not exist. □

Discussion A pythogorean quadruple is an ordered 4-tuple (x, y, z, w) such that $x^2 + y^2 + z^2 = w^2$. Suppose that $x^2 + y^2 = k$ and $w = z + \alpha$ then $k + z^2 = (z + \alpha)^2$. This implies $z = (k - \alpha^2)/2\alpha$. Notice that if k is even then α must be even and if k is odd then α must be odd for $z \in Z$; If k is even then it has to be integer multiple of 2α; $k > \alpha^2$ is necessary for z to be positive. In order to construct distance graphs whose distance set consists of pythagorean quadruples it is pertinent to generate them. We mainly show interest in generating primitive pythagorean quadruples.

Suppose that x is even and y is odd (alternately one can let x as odd and y as even). In this instance it is clear that k is odd and hence α is odd. If x and y have common factors p_j, $j = 1$ to n then $k = \prod_{j=1}^{n} p_j^{m_j} \prod_{j=1}^{N} q_j^{s_j}$ and $\alpha = \prod_{j=1}^{n} p_j^{r_j} \prod_{j=1}^{N} q_j^{t_j}$ where m_j, s_j, r_j and t_j all in Z for all j. $z = (k - \alpha^2)/2\alpha$ becomes $z = \frac{1}{2}\left(\prod_{j=1}^{n} p_j^{m_j - r_j} \prod_{j=1}^{N} q_j^{s_j - t_j} - \prod_{j=1}^{n} p_j^{r_j} \prod_{j=1}^{N} q_j^{t_j}\right)$. So with $k > \alpha^2$, t_j can have all integral values from 0 to s_j so that α takes the above form with either $r_j = 0$ or $r_j = m_j$ for all j. For example, if $x = 12$, $y = 15$ then $k = 369 = 3^2 \times 41$. Note that we have either $\alpha = 1$ or 3^2 and not 41 as $41^2 > k$. So $z = (369 - 1)/2 = 184$ and $w = 185$ with $\alpha = 1$ or $(369 - 81)/18 = 16$ and $w = 25$ with $\alpha = 3^2$. So the primitive quadruples for $x = 12$, $y = 15$ are (12, 15, 184, 185) and (12, 15, 16, 25). Similarly for $x = 210$, $y = 135$ we get four different pythagorean quadruples (210, 135, 1162, 31163), (210, 135, 3458, 3467), (210, 135, 1234, 1259) and (210, 135, 26, 251).

Suppose that x and y are both even. Here k should be an even number and hence it follows that s is even. The arguments as in the above instance applies with the exception that some power of 2 arise here. That is $k = 2^m \prod_{j=1}^{n} p_j^{m_j} \prod_{j=1}^{N} q_j^{s_j}$ and $\alpha = 2^r \prod_{j=1}^{n} p_j^{r_j} \prod_{j=1}^{N} q_j^{t_j}$. Now $z = (k - \alpha^2)/2\alpha$ implies $z = 2^{m-r-1}\left(\prod_{j=1}^{n} p_j^{m_j - r_j} \prod_{j=1}^{N} q_j^{s_j - t_j} - 2^{r-1} \prod_{j=1}^{n} p_j^{r_j} \prod_{j=1}^{N} q_j^{t_j}\right)$. Here the stipulations for getting primitive

quadruples are same with an additional stipulation that $r = 1$ or $r = m - 1$. So the values of α are $\alpha = 2^1(p_1^{r_1} \dots p_n^{r_n})(q_1^{t_1} \dots q_N^{t_N})$ or $\alpha = 2^{m-1}(p_1^{r_1} \dots p_n^{r_n})(q_1^{t_1} \dots q_N^{t_N})$. t_j can take any integer value from 0 to s_j when $k > \alpha^2$ and $r_i = 0$ or n_i. For instance, if $x = 6 = 2 \times 3$, $y = 30 = 2 \times 3 \times 5$ then $k = 2^3 3^2 13 = 936$. The various quadruples that results out are: (6, 30, 233, 235), (6, 30, 115, 119), (6, 30, 17, 35), (6, 30, 5, 31).

Suppose both x and y are odd let $x = 2a + 1$ and $y = 2b + 1$ where $a, b \in Z^+$. Then $k = x^2 + y^2 = (2a + 1)^2 + (2b + 1)^2$. So $k = 4(a^2 + b^2) + 4(a + b) + 2$. k even implies α should be even but k has to contain a factor of 4 to be an integral multiple of 2α. However k is a multiple of 2 only. Hence we infer that no set of quadruples can be got if both x and y are odd.

If D is a set of positive integers and $r \in Z^+$. If $D/gcd(D)$ contains no multiple of r then $\chi(G(Z, D)) \leq r$. Suppose $gcd(D) = 1$ then D contains no multiple of r. In this case we can color the vertices of G with colors from $0, 1, \dots, r - 1$ by allotting to every integer i the color corresponding to the residue class of i modulo r. Two integers will be assigned to the same color only when they differ by a multiple of r. As no multiple of r lies in D it results in a proper coloring of $G(Z, D)$. The converse of this is also true when $r = 2$. This yields as a characterization of the bipartite distance graphs. That is $G(Z, D)$ is bipartite when $D/gcd(D)$ contains no multiple of 2. It is known that if $D \subseteq P$ with $|D| \geq 2$ then $\chi(G(Z, D)) = 2$ if $2 \notin D$, else $\chi(G(Z, D)) = 3$ or 4. Similarly if $2 \in D$ and $3 \notin D$ then $\chi(G(Z, D)) = 3$; if $\{2, 3, 5\}$ or $\{2, 3, 11, 13\} \subseteq D$ then $\chi(G(Z, D)) = 4$. So we deduce that:

Theorem 2.2. *Let $G(Z, D)$ be a distance graph with $|D| = 4$. Let the distance set D consist of primitive Pythagorean quadruples and 2 is not in D. Then $\chi(G(Z, D)) = 2$.*

The Theorem 2.2 is derived out of the observation of the nature of primitive pythagorean quadruples. None of the quadruples have 2 in it. if $r \geq 3$ then it is conjectured that the determination of whether $\chi(G(Z, D)) \leq r$ for finite sets D is NP complete.

3. Motivation for Studying Prime Distance Graphs

Naturally we are attracted to understand the spacing pattern of sequences of numbers. Examples of such sequences are prime numbers, the ordinates of zeros of the Riemann-zeta function, energy levels of large nuclei, the fractional parts of $\sqrt{n}$ for $n \leq N$ etc. From time immemorial a lot of people have spent their energy to find out a procedure submerged in the sequence of prime numbers. Euclid ascertained the infinitude of prime numbers without knowing their pattern. Euler concentrated on the zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$, $s \in N$ and showed that for all $s \in N$, it is connected to prime numbers by $\zeta(s) = \sum_{p \text{is prime}} p^s/(p^s - 1)$. Riemann extended ζ to C, the set of complex numbers and solved the problem of studying prime numbers using Euler product formula and gave a formula for counting the number of primes in $[1, N]$ through the statistical distribution of primes and a lower order oscillatory correction term, determined by the solutions of $\zeta(s) = 0$, $s \in C$. He further conjectured that all the solutions of this equation with $Re(s) > 0$ are of the form $s = (1/2) + b_i$, $b \in R$. Further there are infinitely many zeros with $Re(s) < 0$ and $Im(s) = 0$. This is the famous yet to be resolved 'Riemann hypothesis'. If this hypothesis is true, then it would imply only approximately the distribution of primes which probably could be random. Suppose the hypothesis is false then one can conclude that prime numbers follow some pattern whose identification procedure yet to be unearthed. In [32], Matteo Arpe excellently outlined the

procedure underlying the pattern of prime numbers.

I. Bilingsely [5] defined the set of random walks with the help of decomposition of integers into the primes. In [31] MarekWolf conceived simpler random walks on the primes, through special families of primes. This has sparked interest in us to probe the prime distance graphs. Let G be a finite graph with vertex set $V(G) = \{u_1, \ldots, u_s\}$ and let $n \in Z^+$. We say that G is representable modulo n if there exist distinct integers $x_1, \ldots, x_s$, $0 \leq x_i < n$ such that $x_i - x_j$ is relatively prime to n if and only if $(u_i, u_j) \in E(G)$. We call $\{x_1, \ldots, x_s\}$ a representation of G modulo n. The smallest n such that G is representable modulo n is called the representation number of G and is denoted by $rep(G)$ [1]. It is easy to note that if G is representable modulo n, then $w(G)$ is less than or equal to the smallest prime divisor of x. This can be improved to $\chi(G)$, the chromatic number of G. If p is the smallest prime such that $\chi(G) \leq p$, then there exists a natural number n such that G is representable modulo n and p is the smallest prime divisor of n. To see this, if $\{x_1, \ldots, x_s\}$ is a representation modulo t for G and if p does not divide t then one can construct a representation modulo $n = tp$. Let $f(u_i)$ be the color of vertex u_i represented as an integer between 0 and $p-1$. Let y_i be the unique solution of $y_i t = f(u_i) - x_i (\text{mod } p)$. Then $\{x_1 + y_1 t, \ldots, x_s + y_s t\}$, where each $x_i + y_i t$ is reduced to modulo p_t is a representation of G modulo $n = pt$. Form a distance graph $G(A, D)$ where $D = \{x, y, x+y, y-x\}$ with $1 \leq x < y$ and $(x, y) = 1$. It is easy to see that $\chi(G(A, D)) = 4$ if x and y have different parity. The $rep(G)$ is the smallest integer a such that $(8k+a, 4k+2) = 1$, $(8k+a, 4k+3) = 1$, $(8k+a, 8k+5) = 1$ and $(8k+a, 1) = 1$ if $x = 4k+2$ and $y = 4k+3$. That is, the smallest a such that $(8k+a, 128k^3 + 240k^2 + 148k + 30) = 1$. If x is $4k$ and $y = 4k+1$ then $rep(G)$ is the smallest integer a such that $(8k+a, 128k^3 + 48k^2 + 4k) = 1$.

II. Given a graph $G = (V, E)$, color the vertices with two colors such that the number of edges with equi colored end vertices is a minimum. We denote this minimum by $\theta(G)$ [16]. Erdos in [16] showed that for any (p, q) graph G, $\theta(G) \leq \lfloor q/2 \rfloor$. Note that in a coloring of G of above type, each vertex is adjacent to at least as many vertices of the opposite color as of the same color. This is because, otherwise, we will be able to reduce the number of edges with equi colored end vertices by changing its color.

Theorem 3.1. *If* $t = \lfloor (1/2)(\chi - 1) \rfloor$ *then* $\theta(G) \leq tq/(2t+1)$.

Proof. Suppose that χ is even, say, $\chi = 2t+2$. Color $V(G)$ properly with χ colors $1, 2, \ldots, \chi$. Let the number of vertices in color class i be p_i $(i = 1, 2, \ldots, \chi)$ and let the number of edges between classes i and j be $q_{i,j}$. Now amalgamate the color classes into two groups I and II each being the union of $t+1$ original color classes and color the vertices of I with color a and the vertices of II with the color b. The number of edges with equi colored end vertices is now $\sum_{i,j \in I} q_{i,j} + \sum_{i,j \in II} q_{i,j}$. Note that the groups I and II can be formed in $\frac{1}{2}\binom{2t+2}{t+1}$ ways. Therefore the total number of edges with equi colored end vertices over all possible ways of forming I and II is $\sum q_{ij} \binom{2t}{t-1}$, as there are $\binom{2t}{t-1}$ cases where colors 1 and 2 are in the same class etc. Now as $q = \sum_{i,j} q_{i,j}$, the average number of edges with equi colored end vertices after forming I and II is $q \binom{2t}{t-1} \Big/ \binom{1}{2}\binom{2t+2}{t+1}$ which reduces to $tq/2t+1$. So $\theta(G) \leq qt/(2t+1)$. Suppose that χ is odd, say, $\chi = 2t+1$. Color $V(G)$ properly with χ colors $1, 2, \ldots, \chi$. Let the number of vertices in color class i be p_i

$(i = 1, \ldots, \chi)$ and let the number of edges between classes i and j be q_{ij}. Now amalgamate the color classes into two groups I and II such that I is a union of $\left\lfloor \frac{2t+1}{2} \right\rfloor (= t)$ and II is a union of $\left\lceil \frac{2t+1}{2} \right\rceil (= t+1)$ original color classes. Allot the colors a, b to the groups I and II respectively. The number of edges with equi colored end vertices is now $\sum_{i,j \in I} q_{i,j} + \sum_{i,j \in II} q_{i,j}$. Note that the two groups can be formed in $\binom{2t+1}{t} + \binom{2t+1}{t+1}$ ways. Therefore the total no. of edges with equi colored end vertices over all possible ways of forming I and II is $\sum q_{ij} \binom{2t-1}{t-2} + \sum q_{ij} \binom{2t-1}{t-1}$, as there are $\binom{2t-1}{t-2}$ and $\binom{2t-1}{t-1}$ cases respectively where colors 1 and 2 are in the same class etc. Now as $q = \sum_{i,j} q_{i,j}$ the average no. of edges with equi colored end vertices after forming I and II is $\dfrac{q\binom{2t-1}{t-2}}{\binom{2t+1}{t}} + \dfrac{q\binom{2t-1}{t-1}}{\binom{2t+1}{t+1}}$ which reduces to $qt/2t+1$. So $\theta(G) \leq qt/2t+1$. □

Twin Primes

Prime numbers $p \geq 5$ are known to be of the form $6m \pm 1$. An ordinary twin prime occurs when both $6m \pm 1$ are prime. If $3(2m+1) \pm 2$ is a twin prime pair for some odd $2m+1$ then $2m+1$ is called its twin 4-rank. An odd number $2m+1$ is a non-rank if $3(2m+1) \pm 2$ are not both prime. Odd positive integers ≥ 3 consists of twin 4 and non-ranks only. (Since 2, 3 are not of the form $6m \pm 1$, they are excluded as primes). To construct twin 4-prime sieve one can proceed as follows. Let x be real. Then $A(x)$ is the integer closest to x. If $p \geq 5$ is prime, then $A(p/6) = (p-1)/6$ if $p \equiv 1 (\text{mod } 6)$ and $A(p/6) = (p+1)/6$ if $p \equiv -1 (\text{mod } 6)$. Note that if $p \equiv -1 (\text{mod } 6)$ is prime and $p-4$ is prime, then $((p+1)/3) - 1$ is a twin-4 rank. If $p \equiv 1$ (mod 6) and $(p+4)$ is prime, then $((p-1)/3) + 1$ is a twin-4 rank. This is because $3\left(\left(\frac{p+1}{3}\right) - 1\right) \pm 2 = (p, p-4)$ and $3\left(\left(\frac{p-1}{3}\right) + 1\right) \pm 2 = (p, p+4)$. One can observe that the arithmetic progressions $3.5(2n+1) \pm 2$, $3[5(2n+1)+2] \pm 2$, $3[5(2n+1)-2] \pm 2$, $3[5(2n+1)+4] \pm 2$, $3[5(2n+1)-4] \pm 2$ contain all twin-4 prime pairs. This is because when we remove all pairs $3.5(2n+1) + 12 \pm 2$, $3.5(2n+1) - 12 \pm 2$ from the above progressions then the resulting numbers will be twin-4 prime pairs. In a similar manner one can observe that the arithmetic progressions $3[5.7(2n+1) + c] \pm 2$, $n \geq 0$ contain all twin-4 pairs ≥ 103, where $c \in \{0, 2, 8, 12, 20, 22, 28, 30, 40, 42, 48, 50, 58, 62, 68\}$. Zvi Retchiman Konigsberg in [50] have given some generator and verification algorithms for twin primes.

Twin Prime Conjecture (TPC): There are infinitely many pairs of primes that differ by 2.

We know that there are infinitely many primes and also arbitrarily long gaps between primes. There is also prime number theorem, which tells us that for very large values of x, the number of primes smaller than x is closely approximately $x/ln\ x$. But all these things do not convey much about twin primes. But, finally after a long wait, we come to know that there is some finite number with the property that there are infinitely many pairs of primes that differ by no more than that number. The person who revealed this wonderful result was Yitang Zhang. Everyone is curious to know what is that finite number? According to TPC, the number should be two. Zhang's result shows that there is some number N smaller than 70 million such that there are infinitely many pairs of primes that differ by N. The first pairs of twin primes are: (3, 5) (5, 7), (11, 13), (17,

19), ... with 5 being the only prime in two pairs. Let $\pi_2(x)$ denote number of primes p, not bigger than x such that $p+2$ is also a prime. Then we have $\pi_2(10) = 2$; $\pi_2(11) = 3$, $\pi_2(17) = 4$ etc. The following Table 1 shows the various values of $\pi_2(x)$ for difference powers of 10.

Table 1. Various values of $\pi_2(x)$

$x:$	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}
$\pi_2(x):$	2	8	35	205	1224	8169	58980	440312	3424506	27412679

Green-Tao Theorem: Primes contain arbitrarily long arithmetic progressions.

Beautiful characterizations exist for TPC in the literature. Prominent among them is the one by congruence relations. Let $n > 1$ be an integer. Then n is a prime if and only if $(n-1)! + 1 \equiv 0 \pmod{n}$ if n is a prime and $\equiv 1 \pmod{n}$ otherwise as per famous Wilson's Theorem. Using this Clement has obtained the following result in [10] which characterizes a pair of twin primes.

Clement's Theorem on twin primes: Let $n > 1$ be an integer. Integers n and $n+2$ are both primes, if and only if $4[(n-1)! + 1] + n \equiv 0 \pmod{n(n+2)}$.

4. PDG and PDL

Alternatively Joshua D. Laison et al. in [22] call a graph G, a PDG if there exists a 1-1 labeling $f : V(G) \to Z$ such that for any two adjacent vertices u and v, the integer $|f(u) - f(v)|$ is a prime. f is refereed as a Prime Distance Labelling (PDL) of G. So according to them G is a prime distance graph if and only if there exists a prime distance labeling of G. Note that in a PDL the labels on the vertices of G must be distinct but not the labels on the edges. For instance, the path graph $P_n = u_1u_2 \dots u_{n+1}$ is a PDG with PDL $f : V(P_n) \to Z$ defined by $f(u_i) = 3i - 3$ for $1 \le i \le n+1$. Likewise, every bipartite graph is a PDG. To see this, it is enough to exhibit a PDL of a complete bipartite graph $K_{m,n}$ as every subgraph of a PDG is a PDL. According to Green-Tao's Theorem, for any positive integer k there exists a prime arithmetic progression of length k. That is there is an arithmetic sequence of $m+n-1$ primes $p-(m-1)k$, $p-(m-2)k$, ..., $p-k$, p, $p+k$, ..., $p+(n-2)k$, $p+(n-1)k$. Let U and V be the partite sets of $G = K_{m,n}$ with $|U| = m$ and $|V| = n$. Assign to the vertices of V the labels p, $p+k$, ..., $p+(n-1)k$, and to the vertices of U the labels $0, k, 2k, \dots, (m-1)k$. Then differences between the labels of vertices of U and the labels of vertices of V are all of the form $p + tk$ with $t \in \{-(m-1), -(m-2), \dots, -1, 0, 1, \dots, n-2, n-1\}$ and each such $p + tk$ is a prime. In other words, every 2-chromatic graph is a PDG. But note that not every 3-chromatic graph is a PDG as the complete tri equipartite graph with three vertices on each part is not a PDG. By a Dutch Windmill graph, we mean the graph, $S_{1,2n}$ with central vertex u_0 and leaves $u_1, u_2, \dots, u_{2n}$, with an edge between each consecutive pair of vertices u_{2k-1} and u_{2k}, $1 \le k \le n$. So $S_{1,2n}$ has n copies of C_3 joined at the common vertex u_0. It was showed in [22] that, every Dutch Windmill graph is a PDG if and only if the TPC is true. Now that TPC is shown to be at most true by a result of Zhang, we conclude that a Dutch Windmill graph is most probably, a PDG.

Lemma 4.1. *[22] Every bipartite graph is a prime distance graph.*

Theorem 4.1. *Let $\mathcal{G}$ be the class of all distance graphs $G(Z,D)$ whose distance set D is a subset of primes with $\chi(G(Z,D)) = 2$. Then such class of graphs are prime distance graphs.*

Proof. Let G be any one of the members of $\mathcal{G}$. As $\chi(G) = 2$ it follows that G is bipartite. So by Lemma 4.1 it admits a PDL and hence is a PDG. □

Proposition 4.1. *K_5 is not a prime distance graph.*

Proof. Let $V(K_5) = \{u_1, u_2, u_3, u_4, u_5\}$. Let f be any PDL. $f(u)$ cannot be odd for all $u \in V(K_5)$. This is because, each of the 5(5-1)/2 = 10 possible labels $|f(u_i) - f(u_j)|$ is even for distinct pairs i, j and 2 is the only even prime. The same reasoning precludes the possibility of 4 or 3 odd vertex labels for K_5. So there can be at most two odd vertex labels for K_5. Without loss of generality assume that $f(u_1) = 2r + 1$ and $f(u_2) = 2s + 1$ for some $r, s \in Z$. Then the other three labels must be $2x, 2y, 2z$ for some $x, y, z \in Z$. But this means among these even labels there occurs 3 labels of the form $|f(u_i) - f(u_j)|$ for some distinct i, j with all of them as even numbers, a contradiction, as 2 is the only even prime number. Therefore f cannot be a prime distance labeling. □

Theorem 4.2. *Any graph G with $V(G) \subseteq Z$ and $\chi(G) \geq 5$ do not have a prime distance labeling.*

Proof. If there exists a PDL for all those G with $\chi(G) \geq 5$ then collect all the prime edge labels and call it a set D. Clearly $G(Z,D)$ is a PDG. As $G(Z,D) \subset G(Z,P)$ and χ is a monotone function, the inequality $5 \leq \chi(G(Z,D)) \leq \chi(G(Z,P)) = 4$ results in a contradiction. If $V(G) = Z$, then the similar argument holds good. □

Proposition 4.2. *The graph G shown in Figure 1 has no PDL.*

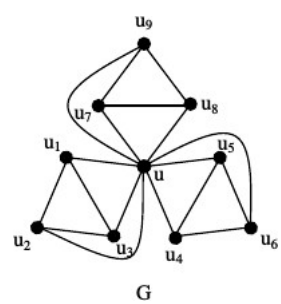

Figure 1.

Proof. Suppose that G has a PDL $f : V(G) \to Z$. Then f can assign either an odd integer label or an even integer label to the vertex u. Without loss of generality assume that $f(u)$ is odd. If f assigns only an odd integer label to all the vertices u_i, $1 \leq i \leq 9$, then it gives raise to more than one even edge label for G. That is, we have for $|f(u_1) - f(u_2)|$, $|f(u_2) - f(u_3)|$ and $|f(u_3) - f(u_1)|$ all even edge labels. But all of them cannot be 2. This is because $u_1u_2u_3u_1$ is a cycle of length 3 and three consecutive odd integers as labels for vertices can produce at most two even edge labels among them of length 2. So at least one vertex from each of the sets of vertices $\{u_1, u_2, u_3\}$; $\{u_4, u_5, u_6\}$ and $\{u_7, u_8, u_9\}$ must receive an even label. Assume without loss of generality that u_1, u_4, u_7 are the respective vertices with an even label under f. This means we have nine even

edge labels among the edges (u_2, u_3), (u, u_2), (u, u_3), (u_5, u_6), (u, u_5), (u, u_6), (u_8, u_9), (u, u_8), (u, u_9). But then this is also not possible, as uu_2u_3u; uu_5u_6u and uu_8u_9u are cycles of length 3. Next we suppose that there are two even labels in each of the sets $\{u_1, u_2, u_3\}$; $\{u_4, u_5, u_6\}$ and $\{u_7, u_8, u_9\}$. Then these two labels along with u constitutes a cycle of length 3 and the same argument applies. Finally if all of u_i's, $1 \leq i \leq 9$ are assigned even labels under f then also the same argument applies. The case that $f(u)$ is even can be disposed off in a similar manner. □

Note 1: Figure 2 shows some 3-regular graphs with PDL.

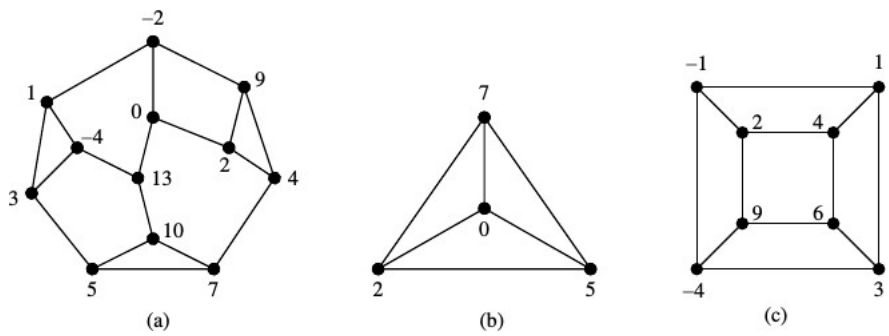

Figure 2.

Problem 4.1. *Prove or Disprove: A k-regular connected graph for $k \geq 4$ admits a PDL?*

Problem 4.2. *Prove or Disprove: A connected k-regular planar graph G for $k \geq 4$ admits no PDL.*

Discussion: We know that for $n, k \in Z^+$ with $k < n$, there exists a k-regular graph on n vertices if and only if nk is even. As the minimum degree of a planar graph is at most 5, we deduce that $0 \leq k \leq 5$. If $k = 0$ then $G \cong nK_1$ and trivially it has a PDL. If $k = 1$ then $G \cong (n/2)K_2$ if $n = |V(G)|$ and $n \equiv 0 \pmod 2$. In this case also G trivially admits a PDL as the ith copy of K_2 bears for its end vertices the label $(4i-3, 4i-1)$ for any odd i and the label $(4i-2, 4i)$ for any even i. If $k = 2$ then $G \cong C_n$. In [22] it was shown that cycles admit PDL. If $k = 3$, and $n = 4$ then we have $G \cong K_4$ and a PDL for the same is shown in Figure 3. Next suppose that G is a 3-regular graph on $2n$ vertices. Let $x_1x_2x_3$ be a path of length three on its outer boundary. Introduce a vertex y_1 on the edge x_1x_2 and a vertex y_2 on the edge x_2x_3. Now join y_1 and y_2 so that the edge joining y_1 and y_2 lies outside the bounded region of G. Call the new graph G^*. Clearly G^* is a 3-regular planar graph on $2n+2$ vertices. Now if G has a PDL then G^* also has a PDL. Figure 4 shows a PDL of only the extended portion of G^* from G. However one should note that the vertices x_1, x_2, x_3 and x_4 and the corresponding graph which forms a part of G should be subjected to a little modification on their vertex labels shown in Figure 4(c) to ensure compatibility with original PDL of G. For $k = 4$, let $n = 2t$, $t \geq 3$. In this case we construct a class of 4-regular planar graphs as follows: First pick a cycle on $2t$ vertices, $C_{2t} = \{u_1, u_2, \ldots, u_{2t}, u_1\}$. Then add two more cycles to C_{2t} with $C_1 : u_1u_3 \ldots u_{2t-1}u_1$ and $C_2 : u_2u_4 \ldots u_{2t}u_2$ with one in the interior region and the other in the exterior region. Call the resulting graph as G_{2t}.

Problem 4.3. *The graph G_{2t} admits no PDL.*

Note 2: In [22] it was pointed out that every 2-chromatic graph that has a PDL is a PDG but not every 3-chromatic graph that has a PDL is a PDG. Interestingly we note that the graph G_8 is a 4-chromatic graph

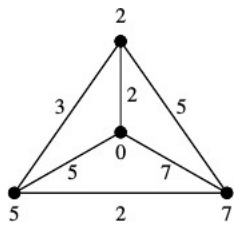

Figure 3.

and hence we conjecture that not every 4-chromatic graph is a PDG. For $k = 5$, if we want a 5-regular planar on n vertices then as nk is even it follows that n is even. Further we know that for a planar graph G on n vertices and m edges, $m \leq 3n - 6$. So by first theorem on Graph Theory we have $5n \leq 6n - 12$. That is, $n \geq 12$. Now consider a 5-regular graph on 12 vertices as shown in Figure 5. The colouring shown in Figure 5. implies that $\chi(G) = 4$.

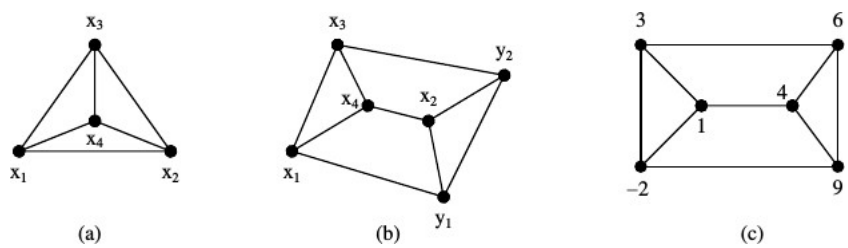

Figure 4.

Problem 4.4. *Prove or Disprove: There exists a 5-regular planar graph that admits no PDL*

The need for studying planar graphs come surprisingly from chemistry. For instance transition metal clusters can be modelled by 3-connected planar graphs. These clusters disallow vertices of degree 5 or more as their presence in them gives raise to angular strain. Consider the octahedron graph shown in Figure 6. We know that if $H \subseteq G$ and G has a PDL then H also has a PDL. In other words, if H has no PDL then G also has no PDL. Hence as the Wheel graph $W_5 = C_4 \vee K_1 \subseteq G$, if W_5 has no PDL then graph G also has no PDL.

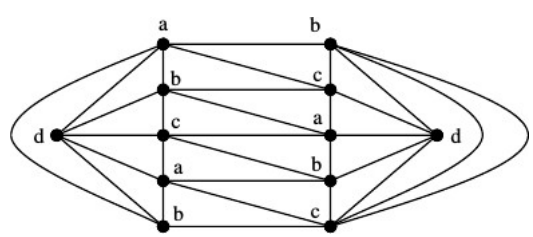

Figure 5.

Theorem 4.3. *Any graph G which contains $W_5 = C_4 \vee K_1$ as an induced subgraph admits no PDL.*

Proof. It is enough to prove that W_5 has no PDL. Let u denote the vertex K_1 of W_5. Let $u_1u_2u_3u_4u_1$ be the cycle graph of W_5. If f is a PDL of W_5 then $f(u)$ can be either odd or even. Suppose that $f(u)$ is odd. Then the following cases arise.

Case 1 $f(u_i)$ is odd for $i = 1, 2, 3, 4$. In this case all edges must have the same prime label 2 the only even prime label. But then a K_3 graph induced by uu_1u_2u cannot have the label 2 for all its edges as there are only 2 different sequence of three consecutive odd numbers possible.

Case 2 $f(u_i)$ is odd for $i = 1, 2, 3$ and $f(u_4)$ is even

In this case also we have the K_3 graph uu_1u_2u with label 2 for all its edges. In a similar manner we can dispose off other combinations of 3 odd labels and 1 even label around u.

Case 3 $f(u_i)$ is odd for $i = 1, 2$ and $f(u_2)$ and $f(u_4)$ are even

In this case uu_1, uu_2 are consecutive odd integers. So if $f(u) = 2k - 1$ then $f(u_1) = 2k - 3$ or $2k + 1$ and $f(u_3) = 2k - 3$ or $2k + 1$. If $f(u_1) = 2k - 3$ then $f(u_3) = 2k + 1$. Now if $f(u_2) = 2m$ for some $m \in Z$, then $2(m-k)+3$, $2(m-k)+1$, $2(m-k)-1$ are three consecutive numbers. When we reduce them by congruence modulo 3 then one among them must be a multiple of 3. We know that there exists in such a case a unique triple 3, 5, 7. Therefore the edge u_2u_1 will have the label 3, the edge u_2u will have the label 5 and the edge u_2u_3 will have the label 7. In such a occurrence the label of u_2 will turn out to be $2k - 6$. But then we have similar situation for the edges u_4u_1, u_4u, and u_4u_3 requiring for the vertex u_4 the same label $2k - 6$ of u_1, a contradiction. In a similar manner we can dispose off other combinations of 2 odd labels and 2 even labels around u.

Case 4 $f(u_1)$ is odd and $f(u_2)$, $f(u_3)$ and $f(u_4)$ are all even.

This case can be disposed off with a similar argument as in Case 1 and also other combinations of 1 odd and 3 even labels around u.

Case 5 $f(u)$ is even

As before we can produce the same argument for all the 4 possible cases and their sub cases.

□

Note 3: Suppose a connected 4-regular planar graph G contains one of the configurations of types shown in Figure 7 wherein each of them has a wheel W_5 as an induced subgraph admits no PDL

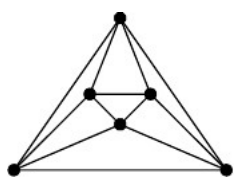

Figure 6.

Discussion: Consider the graph G shown in Figure 8. We know that if G is planar and has order 6 then maximum number of edges it can have is at most $3 \times 6 - 6 = 12$. As G has twelve edges it is easy to see that it is a maximal planar graph. Also we know that if G is a maximal planar graph of order $n \geq 4$ and size m containing n_i vertices of degree i for $3 \leq i \leq \Delta$, then $3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta$. Now as G is 4-regular $n_i = 0$ for all $i \neq 4$. So $2n_4 = 12$ and $n_4 = 6$. This observation reveals that G is infact a unique 4-regular, planar graph. Since it has a W_5 as an induced subgraph by Theorem 4.3 we conclude that

Theorem 4.4. *A unique 4-regular, maximal planar graph has no PDL.*

Discussion: In [6] the authors have devised a method to generate all 3-connected 4-regular planar graphs from the octahedron graph. They pointed out if G a 4-regular, planar graph contains one of the configurations of type A, B, C, D, E, F shown in Figure 9 then by defining the following reduction operations applicable to the

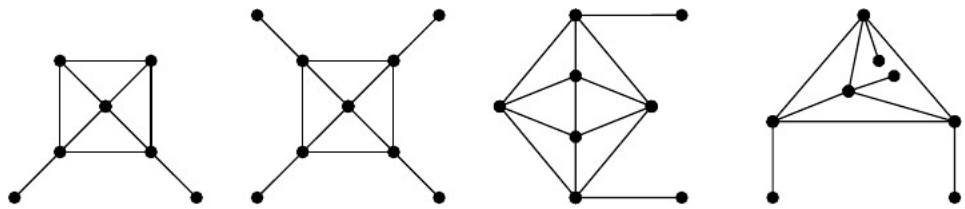

Figure 7.

respective configurations, it can be transformed to a connected 4-regular planar graph of a lower order. The operations are:

ϕ_A : delete the vertex u and add the edges u_1u_3 and u_2u_4

ϕ_B : delete u, v, w and add edge u_1u_2. Also add a new vertex x and add edges xu_1, xu_2, xu_3, xu_4

ϕ_C : delete u_1, u_2, u, u_3, u_4 and add a new vertex x with edges xu_5, xu_6, xu_7, xu_8.

ϕ_D : delete u_1, u_2, u_3, u_4, u_5, u_6 and add the edge xy

ϕ_E : delete u, v, w, u_1; add a new vertex z with edges zu_2, zu_3, zx, zy

ϕ_F : delete u_1, v_4 and add the edges v_2u_2, v_3u_3, v_1u_4.

When the graphs A, B, C, D, E and F undergo the reduction operations then they will look like the one in Figure 10. Notice that all these modified configurations in Figure 10 admits PDL.

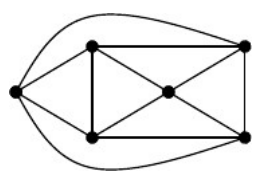

Figure 8.

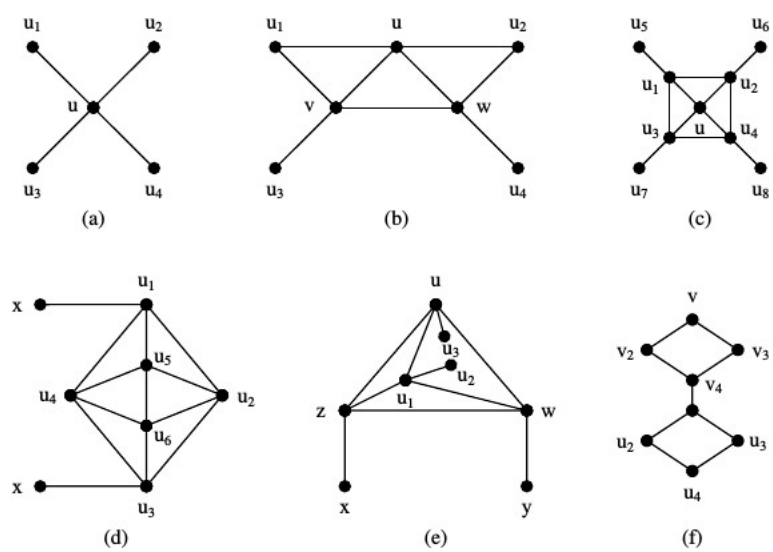

Figure 9.

Proposition 4.3. *Given any infinite number of primes $p_1, p_2, \ldots, p_{|P|}$ there exists a graph G with $p_1, p_2, \ldots, p_{|P|}$*

as the labels for their edges where P denotes the set of all primes.

Proof. Consider the star graph $G = K_{1,|P|}$. Let $f : V(G) \to Z$ be a one-to-one function. Define the vertices of G as : $V(G) = \{u, u_1, \ldots, u_{|P|}\}$. Let $f(u) = 0$ and $f(u_i) = p_i$ for $1 \leq i \leq |P|$. Then it is easy to see that f is a PDL of G. □

Proposition 4.4. *Given any prime p, there exists a graph G with p as a label for all its edges.*

Proof. Consider the path graph $P_{\frac{|Z^+|p}{p}+1} : u_1u_2 \ldots u_{\frac{|Z^+|p}{p}+1}$. Define a 1-1 function $f : V\left(P_{\frac{|Z^+|p}{p}+1}\right) \to Z$ by $f(u_i) = (i-1)p$ for $1 \leq i \leq \frac{|Z^+|p}{p} + 1$. Then it is easy to see that the label of any arbitrary edge (u_j, u_{j+1}) is p for $1 \leq j \leq \frac{|Z^+|p}{p} + 1$. □

Proposition 4.5. *Given any pair of twin primes (p_i, p_{i+1}) there exists a graph G whose edge labels alternate between p_i and p_{i+1}.*

Proof. Consider the path graph P_n, $n \leq |Z|$ with $P_n = u_1u_2 \ldots u_n$. Define a 1-1 function $f : V(P_n) \to Z$ such that $f(u_1) = 0$, $f(u_2) = p_i$, $f(u_3) = 2p_i + 1$, $f(u_{2i+2}) = f(u_{2i}) + (2p_i + 1)$ for $i \geq 1$. $f(u_{2i+3}) = f(u_{2i+1}) + (2p_i + 1)$ for $i \geq 1$. Then it is easy to see that p_i and p_{i+1} alternates as labels for the edges of the path graph. □

Theorem 4.5. *Given any n-tuple of primes $(p_1, p_2, \ldots, p_n)$ with $n \geq 1$ there exists a graph G where the primes $p_1, p_2, \ldots, p_n$ appears as edge labels in order with desired number of repetitions in the same order.*

Proof. On similar lines as in Proposition 4.5. □

Note 4: By Turan's theorem, the maximum number $f(n)$ of edges a graph G on n vertices not having K_4 as an induced subgraph can have is $f(n) = \frac{n^2}{2} + \frac{r(r-3)}{2}$, $r = 1, 2, 3, n \equiv r \pmod 3$. If we take $n = 7$; then $f(n) = 16$. Interestingly the unit distance graph G_1 shown in Figure 11. with 11 edges also called the famous Moser Spindle graph is a prime distance graph with chromatic number 4 is a PDG with integers as vertex set and $D = \{2, 3, 5, 7, 11, 13\}$ as prime distance set. Incidentally, G_1 also admits a PDL as shown in the Figure 11.

In [22] Laison et al. have raised the following open problem "Is there a family of graphs which are PDGs' if and only if Goldbach's conjecture is true?" We answer the above question in affirmative in Theorem 4.6.

Theorem 4.6. *Let G_1 be any connected 2-regular graph with n vertices and G is the graph obtained from G_1 by duplicating an arbitrary vertex of it. Then G admits a PDL if and only if Goldbach's conjecture is true.*

Proof. Let $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $E(G_1) = \{u_iu_{i+1}, u_nu_1 \text{ for } 1 \leq i \leq n-1\}$. Obtain a new graph G from G_1 by duplicating the vertex $u_n \in V(G_1)$. Then $V(G) = V(G_1) \cup \{u'_n\}$ where u'_n is the vertex corresponding to the duplicating vertex $u_n \in V(G_1)$ and $E(G) = E(G_1) \cup \{(u'_n, u_1), (u'_n, u_{n-1})\}$. Define a bijection $f : V(G) \to Z^+$ as follows: $f(u_i) = 2(i-1)$ for $1 \leq i \leq n-1$. Look at the label $2n-4$ of the vertex u_{n-1}. If Goldbach's conjecture is true, then we can express the even number $2n-4$ as the sum of two primes say, $2n-4 = p_1 + p_2$. Now let $f(u_n) = p_1$ and $f(u'_n) = p_2$. Then it is easy to check that f is a PDL of G as $|f(u_i) - f(u_{i+1})| = 2$ for $1 \leq i \leq n-1$; $|f(u_n) - f(u_1)| = p_1$; $|f(u_n) - f(u_{n-1})| = p_2$; $|f(u'_n) - f(u_1)| = p_2$; $|f(u'_n) - f(u_{n-1})| = p_1$. □

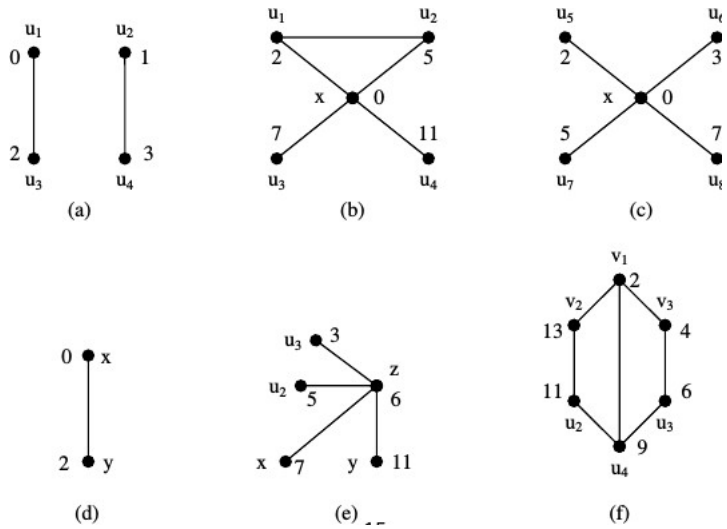

Figure 10.

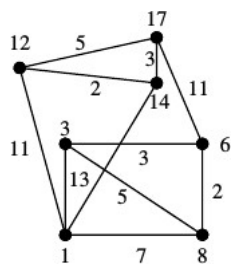

Figure 11.

5. Applications

Problem 5: Efficiently assign radio channels to transmitters at several locations using non-negative integers to represent channels so that close locations receive different channels and channels that are very close to each other are separated by a distance of at least two. Clearly such an assignment would result in non interference of channels with each other. The above said problem can be described analogously in the language of graph theory as follows:

Let $G = (V, E)$ be a simple graph. Given a real number $d > 0$, an $L_d(2,1)$-labeling of G is a non-negative real-valued function $f : V(G) \to [0, \infty)$ such that whenever x and y are two adjacent vertices in V, then $|f(x) - f(y)| \geq 2d$, and whenever the distance between x and y is 2, then $|f(x) - f(y)| \geq d$. The $L_d(2,1)$- labeling number of G is the smallest m such that G has an $L_d(2,1)$-labeling with no label greater than m and is denoted by $\lambda(G, d)$. If f is a $L_d(2,1)$-labeling of G, then we say that $f \in L_d(2,1)(G)$. Define $||f(G)|| = \max\{f(v) : v \in V(G)\}$. Then $\lambda(G, d) = \min ||f(G)||$ where the minimum runs over all $f \in L_d(2,1)(G)$. Now the problem is to minimize the span of an $L_d(2,1)$-labeling. The following observations are made by Griggs and Yeh in [20].

Proposition 5.1. $\lambda(G, d) = d\lambda(G, 1)$

Proposition 5.2. *Let $x, y \geq 0$, $d > 0$ and $k \in Z^+$. If $|x - y| \geq kd$ then $|x' - y'| \geq kd$ where $x' = \lfloor x/d \rfloor d$*

and $y' = \lfloor y/d \rfloor d$.

Using Proposition 5.1 and Proposition 5.2 they obtained the following result.

Theorem 5.1. *Given a graph, there is an* $f \in L_1(2,1)(G)$ *such that* f *is integer valued and* $||f(G)|| = \lambda(G,1)$.

So in view of the above for general d, $\lambda(G,d)$ is attained by some $f \in L_d(2,1)(G)$ whose values are all multiples of d, with $f = d.f'$, where $f' \in L_1(2,1)(G)$ is integral valued.

Theorem 5.2. *Let* G *be a graph with* $\chi(G) = k$ *and* $|V(G)| = \nu$. *Then* $\lambda(G) \leq \nu + k - 2$.

Proof. Since $\chi(G) = k$ we can partition G into $\bigcup_{i=1}^{k} G_i$ where $|V(G_i)| = \nu_i$ and each $V(G_i)$ is an independent set. Let $V_i = V(G_i) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,v_i}\}$ $1 \leq i \leq k$. Now consider the labeling f defined by $f(v_{1,j}) = j - 1$, $1 \leq f \leq \nu_1$. $f(v_{i,j}) = \sum_{t=1}^{i-1} v_t + i + j - 2$, $1 \leq j \leq \nu_i$, for $2 \leq i \leq k$. Then it is easy to see that $f \in L(2,1)(G)$. Hence $\lambda(G) \leq ||f(G)|| = \nu + k - 2$. □

Theorem 5.2 established in [20] reveals the relationship between $L_d(2,1)$-labeling number λ and the chromatic number of a graph G. In view of this we have the following inequality for the λ of a prime distance graph $G(Z,P)$. We know that $\chi(G(Z,P)) = 4$ and hence $\lambda(G(Z,D)) \leq |V(Z)| + 2$. Consider the complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$, with $n_1 + n_2 + \cdots + n_k = \nu$. By Theorem 5.2, $\lambda(K_{n_1,n_2,\ldots,n_k}) \leq \nu + k - 2$ where $k = \chi(K_{n_1,n_2,\ldots,n_k})$. But also note that the distance between any two vertices in $K_{n_1,n_2,\ldots,n_k}$ is at most 2. Therefore the labels must be distinct. Moreover, consecutive labels cannot be used at vertices from different parts. As there are k components, we find that $\lambda(K_{n_1,n_2,\ldots,n_k}) \geq \nu + k - 2$, and hence $K_{n_1,n_2,\ldots,n_k}$ an extreme graph for the graph equation $\lambda(G) = \nu + k - 2$.

References

[1] Anthony B. Evans, Gerd H. Fricke, Meneri Carl C, Mckee Terry A, Manley Perkel. Representations of Graphs Modulo n. J. Graph Theory 1994; 18(8): 801–815.

[2] Barning FJM. On Pythagorean and quasi-Pythagorean triangles and a generation process with the help of unimodular matrices, (Dutch) Math. Centrum Amsterdam Afd. Zuivere Wisk, ZW-011, 1963.

[3] Benoit R. Kloeckner. Coloring distance graphs: a few answers and many questions by hal Id: hal-00821852 https://hal.archives-ouvertes.fr/hal-00821852 Submitted on 13 May 2013.

[4] Bondy JA, Murty USR. Graph Theory with Applications. The Macmillan Press Ltd., (1976).

[5] Billingely. Prime Numbers and Brownian motion. Amer. Math. Monthly 1973; 80: 1099–1115.

[6] Broersma HJ, Duijvestijn AJW, Gobel F. Generating all 3-connected 4-regular graphs from the octahedron graph. Journal of Graph Theory 1993; 17(5): 613–620.

[7] Chang GJ, Chen JJ, Huang K-Ch. Integral distance graphs. J. Graph Theory 1997; 25: 287–294.

[8] Chilakamarri KB. Unit-distance graphs in Minkowski metric spaces. Geom. Dedicata 1991; 37(3): 345–356.

[9] Chromatic number of the plane & its relatives, history, problems and results: an essay in 11 parts, Ramsey theory, Progr. Math., vol. 285, Birkhauser/Springer, New York, 2011, 121–161.

[10] Clement PA. Congruences for sets of primes. The American Mathematical Monthly 1949; 56(1): 23–25.

[11] Datta B, Singh AN. History of Hindu Mathematics, A Source Book, Parts 1 and 2, (single volume), Asia Publishing House, Bombay, 1962.

[12] de Bruijn NG, Erdos P. A colour problem for infinite graphs and a problem in the theory of relations. Indagationes Math. 1951; 13: 369–373. [Exo05] G. Exoo - Q-unit distance graphs, Discrete Comput. Geom. 33 (2005), no. 1, p. 117–123.

[13] (ed.)-Ramsey theory, Progress in Mathematics, vol. 285, Birkhauser/Springer, New York, 2011. DISTANCE GRAPHS 17.

[14] Eggleton RB, Erdos P, Skilton DK. Coloring the real line. J. Combin. Theory (B) 1985; 39: 86–100.

[15] Eggleton RB, Erdos P, Skilton DK. Coloring prime distance graphs. Graphs and Combinatorics 1990; 6: 17–32.

[16] Erdos P. On some extremal problems in graph theory. Israel. J. Math. 1965; 3: 113–116.

[17] Falconer KJ. The realization of distances in measurable subsets covering Rn. J. Combin. Theory Ser. A 1981; 31(2): 184–189.

[18] Harary F. Graph Theory, Addisson-Wesley Publishing Company, Reading, 1969.

[19] Jensen TR, Toft B. Graph coloring problems, John Wiley & Sons, New York, 1995.

[20] Jerrold R. Griggs, Roger K. Yeh. labelling Graphs with a condition at distance 2. SIAM J. Discrete. Math. 1992; 5(4): 586–595.

[21] Johnson Jr PD, Szlam AD. A new connection between two kinds of Euclidean coloring problems. Geombinatorics 2001; 10(4): 172–178.

[22] Josua D, Laison, Colin Starr, Andrea Walker. Finite prime distance graphs and 2-odd graphs. Discrete. Math 2013; 313(20): 2281–2291.

[23] Kahle M. chromatic number of the hyperbolic plane, MathOverflow, 2012, http://mathoverflow.net/questions/86234/

[24] Kak S. The structure of the Rgveda. Indian Journal of History of Science 1993; 28: 71–79.

[25] Kak S. The Asvamedha: The Rite and its Logic, Motilal Banarsidass, Delhi, 2002.

[26] Kak S, Shulba Sutras. In Encyclopedia of India (edited by Stanley Wolpert). Charles Scribner's Sons/Gale, New York, 2005.

[27] Kak S. Early record of divisibility and primality, arXiv:0904.1154

[28] Kak S. Aryabhata's mathematics, arXiv:1002.3409

[29] Kemnitz A, Kolberg H. Chromatic number of integer distance graphs. Discrte. Math 1998; 191: 113–123.

[30] Lec 6 — MIT 6.042J Mathematics for Computer Science, Fall 2010 — Video Lecture.

[31] Marek Wolf, Random walk on the prime numbers, *Physica A* (1998), 335–344.

[32] Matteo Arpe. The Rule behind the occurrence of prime numbers. Italian Journal of Pute and Applied Mathematics 2008; N-23: 1–16.

[33] McCullough D. Height and excess of Pythagorean triples. Mathematics Magazine. 2005; 78: 26–44.

[34] Neugebauer O, Sachs A. Mathematical Cuneiform Texts, New Haven, CT., 1945.

[35] O'Conner JJ, Robertson EF, Baudhayana. History of Mathematics Project. http://wwwhistory.mcs.st-and.ac.uk/ history/Biographies/Baudhayana.html

[36] Rao TRN, Kak S. Computing Science in Ancient India, Munshiram Manoharlal, New Delhi, 2000.

[37] Roberts FS. From Garbage to Rainbows: Generalizations of Graph Coloring and their Applications. In Alavi, Y., Chartrand, G., Oellermann, O.R. , and Schwenk, A.J. (eds.), Graph Theory, Combinatorics, and Applications. Wiley, New York, 1991.

[38] Seidenberg A. The origin of mathematics. Archive for History of Exact Sciences 1978; 18: 301–42.

[39] Sen SN, Bag AK. The Śulbasūtras. Indian National Science Academy, New Delhi, 1983.

[40] Shelah S, Soifer A. Axiom of choice and chromatic number of the plane. J. Combin. Theory Ser. A 2003; 103(2): 387–391

[41] Simmons GJ. The chromatic number of the sphere. J. Austral. Math. Soc. Ser. A. 1976; 21(4): 473–480.

[42] Soifer A. The mathematical coloring book, Springer, New York, 2009, Mathematics of coloring and the colorful life of its creators, With forewords by Branko Grunbaum, Peter D. Johnson, Jr. and Cecil Rousseau.

[43] Srinivasiengar CN. The History of Ancient Indian Mathematics, The World Press, Calcutta, 1967.

[44] Supriya Mohanty, Mohanty SP. Pythagorean numbers. Fibonacci Quaterly 1990; 28: 31–42.

[45] Van der Waerden BL. Science Awakening, Oxford University Press, 1971.

[46] Voigt M. Coloring of distance graphs. Ars Combin 1999; 52: 3–12.

[47] Voigt M, Walther H, Chromatic number of iprime distance graphs. Discrete. App. Math 1994; 51: 197–209.

[48] Walther H. Uber eine spezielle Klasse unendilicher Graphen, Graphenthorie II (Klauss Wagner and Rainer Bodendiek, eds) Bibliographisches Institut Wissenschaftsverlag 1990; 268–295 (German).

[49] Zhu Z. Distance graphs in the real line. Manuscript, 1996.

[50] Zvi Retchiman Konigberg. Generator and Verification Algorithms for prime k-tuples using Binomial Expressions. International Mathematical Forum 2011; 6: 2169–2178.