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On d'Alembert's like functional equation involving an endomorphism

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Abstract: Given an (not necessarily involutive) endomorphism $\varphi : G \to G$ of a group G we find the solutions $f, g : G \to \mathbb{C}$ of the following functional equation

$$f(xy) - f(\varphi(y)x) = 2g(x)g(y), \quad x, y \in G,$$

in terms of characters and additive functions on G. This allows us to solve the more general equation

$$f(xy) + g(\varphi(y)x) = h(x)h(y), \quad x, y \in G,$$

in which $f, g, h : G \to \mathbb{C}$ are the unknown functions.

1. Introduction and notation

Throughout the paper we work in the following framework and with the following notation and terminology. We use it without explicit mentioning.

G is a group with identity element e and the map $\varphi : G \to G$ denotes an endomorphism of G, i.e.,

- $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G$;
- $\varphi(e) = e$;
- $\varphi(x^{-1}) = (\varphi(x))^{-1}$ for all $x \in G$.

A function $A : G \to \mathbb{C}$ is called additive, if it satisfies $A(xy) = A(x) + A(y)$ for all $x, y \in G$.

A character of G is a homomorphism from G into the multiplicative group of non-zero complex numbers. It is well known that the set of characters on G is a linearly independent subset of the vector space of all complex-valued functions on G (see [12, Corollary 3.20]).

By $\mathcal{N}(G, \varphi)$ we mean the vector space of the solutions $\theta : G \to \mathbb{C}$ of the homogeneous equation

$$\theta(xy) - \theta(\varphi(y)x) = 0, \quad x, y \in G.$$

If G is a topological space, then we let $C(G)$ denote the algebra of continuous functions from G into $\mathbb{C}$.

In the papers [2–4] about vibrating strings d'Alembert studied not just what is now called d'Alembert's functional equation, i.e.,

$$g(x + y) + g(x - y) = 2g(x)g(y), \quad x, y \in \mathbb{R}, \tag{1}$$

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in which $g : \mathbb{R} \to \mathbb{R}$ is the unknown function, but also the functional equation

$$f(x+y) - f(x-y) = g(x)h(y), \quad x, y \in \mathbb{R}, \tag{2}$$

in which $f, g, h : \mathbb{R} \to \mathbb{R}$ are unknown functions. For further contextual and historical discussion we refer, e.g., to [1, 5, 6].

Let S be a semigroup, i.e. a set equipped with an associative composition rule $(x, y) \mapsto xy$, and let $\sigma \in Hom(S, S)$ satisfy $\sigma^2 = id$. In [14], Stetkær introduced and solved the following generalization of d'Alembert's functional equation (1)

$$g(xy) + g(\sigma(y)x) = 2g(x)g(y), \quad x, y \in S, \tag{3}$$

where $g : S \to \mathbb{C}$ is the unknown function (he named it a variant of d'Alembert's functional equation). He showed that the general solution of (3) is

$$g = \frac{\chi + \chi \circ \sigma}{2},$$

where $\chi : S \to \mathbb{C}$ is multiplicative.

In [5], Ebanks and Stetkær determined the complex-valued solutions (f, g, h) of the functional equation

$$f(xy) - f(\sigma(y)x) = g(x)h(y), \quad x, y \in S, \tag{4}$$

when S is a group (or a monoid generated by its squares). This equation contains the functional equation (2) whose solutions are known on abelian groups, and a functional equation

$$f(x+y) - f(x+\sigma(y)) = g(x)h(y),$$

studied by Stetkær [13, Corollary III.5] on abelian groups.

In [9], Fadli et al. studied the solutions $g : S \to \mathbb{C}$ of the functional equation

$$g(xy) + g(\phi(y)x) = 2g(x)g(y), \quad x, y \in S, \tag{5}$$

where $\phi : S \to S$ is an endomorphism (i.e. $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in S$). This equation is a generalization of the variant (3) of d'Alembert's functional equation since ϕ need not be involutive. They showed that any solution $g : S \to \mathbb{C}$ of (5) can be expressed in terms of multiplicative functions on S (see [9, Theorem 3.1]).

In [11], Sabour determined the complex-valued solutions (f, g) of the functional equation

$$f(xy) + f(\phi(y)x) = 2f(x)g(y), \quad x, y \in S,$$

when S is a group (or a monoid generated by its squares). This equation, in the case where $\phi^2 = id$, has been solved by Fadli et al. in [7].

The present paper studies similar functional equations, i.e., functional equations with an endomorphism. It gives an extension of the functional equation (4) when $h = 2g$. More precisely, we find the solutions $f, g : G \to \mathbb{C}$ of the functional equation

$$f(xy) - f(\varphi(y)x) = 2g(x)g(y), \quad x, y \in G, \tag{6}$$

in terms of characters and additive functions on G. As an application of this result and [11, Theorem 3.3], we obtain the solutions $f,g,h:G\to\mathbb{C}$ of the more general functional equation

$$f(xy)+g(\varphi(y)x)=h(x)h(y),\quad x,y\in G. \tag{7}$$

Note that the solution of this equation, in the case where $\varphi^2=id$, can be found, e.g., in [1, 8, 10].

2. Solution of equation (6)

In this section, we solve the functional equation (6), namely,

$$f(xy)-f(\varphi(y)x)=2g(x)g(y),\quad x,y\in G,$$

by expressing its solutions in terms of characters and additive functions. The following lemma lists pertinent basic properties of any solution $f,g:G\to\mathbb{C}$ of (6).

Lemma 2.1. *If the pair $f,g:G\to\mathbb{C}$ satisfies (6), then $f\circ\varphi=f$ and $g\circ\varphi=-g$.*

Proof. Putting $y=e$ in (6) we obtain $0=2g(e)g(x)$ for all $x\in G$, so that $g(e)=0$. Now, letting $x=e$ in (6) we find that $f\circ\varphi=f$.

Replacing (x,y) by $(\varphi(y),x)$ in (6) and then adding the resulting equation to (6) we get that

$$0=2g(x)[g(y)+g\circ\varphi(y)]\quad\text{for all } x,y\in G,$$

which implies that $g\circ\varphi=-g$. □

The main theorem of the present section reads as follows (for the notation $\mathcal{N}(G,\varphi)$ see Section 1).

Theorem 2.1. *The pair $f,g:G\to\mathbb{C}$, where $g\neq 0$, satisfies the functional equation (6) if and only if there exist a character $\chi:G\to\mathbb{C}$ with $\chi\circ\varphi^2=\chi$, a constant $c\in\mathbb{C}\backslash\{0\}$ and a function $\theta\in\mathcal{N}(G,\varphi)$ such that one of the following holds.*

(i) If $\chi\circ\varphi\neq\chi$, then

$$f=\theta+2c^2(\chi+\chi\circ\varphi)\quad\text{and}\quad g=c(\chi-\chi\circ\varphi).$$

(ii) If $\chi\circ\varphi=\chi$, then there exists a non-zero additive function $A:G\to\mathbb{C}$ with $A\circ\varphi=-A$ such that

$$f=\theta+\frac{1}{2}\chi A^2\quad\text{and}\quad g=\chi A.$$

Moreover, if G is a topological group, φ is continuous and $f,g\in C(G)$, then $\chi,\chi\circ\varphi,A,\theta\in C(G)$.

Proof. Making the substitutions (x,yz), $(\varphi(y),\varphi(z)x)$, and $(\varphi^2(z),\varphi(xy))$ in (6), we get respectively

$$\begin{aligned} f(xyz)-f(\varphi(yz)x) &= 2g(x)g(yz),\\ f(\varphi(yz)x)-f(\varphi(\varphi(z)x)\varphi(y)) &= 2g(\varphi(y))g(\varphi(z)x),\\ f(\varphi(\varphi(z)x)\varphi(y))-f\circ\varphi^2(xyz) &= 2g(\varphi^2(z))g(\varphi(xy)). \end{aligned}$$

Summing these three equations and using lemma 2.1 results in

$$g(x)g(yz) - g(y)g(\varphi(z)x) - g(z)g(xy) = 0. \tag{8}$$

Choosing here $x = x_0 \in G$ such that $g(x_0) \neq 0,$ we arrive at

$$g(yz) = g(y)l_1(z) + g(z)l_2(y), \tag{9}$$

where $l_1(z) := \frac{g(\varphi(z)x_0)}{g(x_0)}$ and $l_2(y) := \frac{g(x_0y)}{g(x_0)}$. Using (9) in (8) we get after some rearrangement

$$\begin{aligned} &g(x)g(y)[l_1(z) - l_2 \circ \varphi(z)] + g(x)g(z)[l_2(y) - l_1(y)] \\ &+g(y)g(z)[l_1(x) - l_2(x)] = 0 \quad \text{for all } x, y, z \in G. \end{aligned} \tag{10}$$

Putting here $x = x_0,\ y = x_0,$ we derive the equation

$$\begin{aligned} &g(x_0)^2[l_1(z) - l_2 \circ \varphi(z)] + g(x_0)g(z)[l_2(x_0) - l_1(x_0)] \\ &+g(x_0)g(z)[l_1(x_0) - l_2(x_0)] = 0 \quad \text{for all } z \in G. \end{aligned}$$

From this last equation, we see that $l_1 = l_2 \circ \varphi$. So Eq. (10) becomes

$$g(x)g(z)[l_2(y) - l_2 \circ \varphi(y)] + g(y)g(z)[l_2 \circ \varphi(x) - l_2(x)] = 0$$

for all $x, y, z \in G$. Letting here $x = x_0, z = x_0,$ we obtain $l_2 = l_2 \circ \varphi + \alpha g$ for some constant $\alpha \in \mathbb{C}$. And with this, (9) becomes

$$\begin{aligned} g(yz) &= g(y)l_2 \circ \varphi(z) + g(z)[l_2 \circ \varphi(y) + \alpha g(y)] \\ &= g(y)[l_2 \circ \varphi(z) + \frac{\alpha}{2}g(z)] + g(z)[l_2 \circ \varphi(y) + \frac{\alpha}{2}g(y)]. \end{aligned}$$

This shows that the pair (g, k), where $k := l_2 \circ \varphi + \frac{\alpha}{2} f$, satisfies the sine addition law. Since $g \neq 0$, then lemma 3.4 in [5] tells us that there exist two multiplicative functions $\chi_1, \chi_2 : G \to \mathbb{C}$ such that

$$k = \frac{\chi_1 + \chi_2}{2},$$

and we have only the following two cases.

Case 1: Suppose $\chi_1 \neq \chi_2$. In that case $g = c(\chi_1 - \chi_2)$ for some constant $c \in \mathbb{C}\backslash\{0\}$. Since $g \circ \varphi = -g$ (see lemma 2.1), we obtain

$$\chi_1 \circ \varphi + \chi_1 = \chi_2 + \chi_2 \circ \varphi.$$

Since $\chi_1 \neq \chi_2$, then $\chi_1 = \chi_2 \circ \varphi$ and $\chi_2 = \chi_1 \circ \varphi$ (see [12, Corollary 3.19]). From this we infer that

$$\chi_1 \circ \varphi^2 = \chi_2 \circ \varphi = \chi_1 \quad \text{and} \quad g = c(\chi_1 - \chi_1 \circ \varphi).$$

Note that χ_1 is a character of G since $\chi_1 \neq 0$ ($\chi_1 = 0 \Rightarrow \chi_2 = \chi_1 \circ \varphi = 0$, which contradicts the fact that $\chi_1 \neq \chi_2$). Inserting the form of g into Eq. (6) we arrive at

$$\begin{aligned} f(xy) - f(\varphi(y)x) &= 2g(x)g(y) \\ &= 2c^2[\chi_1(x) - \chi_1 \circ \varphi(x)][\chi_1(y) - \chi_1 \circ \varphi(y)] \\ &= 2c^2[\chi_1(xy) - \chi_1(\varphi(y)x) - \chi_1(y\varphi(x)) \\ &\quad +\chi_1 \circ \varphi(xy)] \\ &= 2c^2[(\chi_1 + \chi_1 \circ \varphi)(xy) - (\chi_1 + \chi_1 \circ \varphi)(\varphi(y)x)]. \end{aligned}$$

Defining $\theta := f - 2c^2(\chi_1 + \chi_1 \circ \varphi)$, we see that $\theta \in \mathcal{N}(G,\varphi)$ and arrive at the solution in (i) with $\chi = \chi_1$.

Case 2: Suppose $\chi_1 = \chi_2 = \chi$. In that case χ is a character and $g = \chi A$ for some non-zero additive function $A : G \to \mathbb{C}$. Since $g \circ \varphi = -g$, we have

$$\chi A + \chi \circ \varphi A \circ \varphi = 0.$$

Since $A \neq 0$, we obtain $\chi = \chi \circ \varphi$ (see [13, lemma II.1]) and hence $A \circ \varphi = -A$. Putting the expression $g = \chi A$ into Eq. (6) we obtain after an elementary computation that

$$\begin{aligned} f(xy) - f(\varphi(y)x) &= 2g(x)g(y) = 2\chi(x)A(x)\chi(y)A(y). \\ &= \frac{1}{2}\chi(xy)[A^2(xy) - A^2(\varphi(y)x)]. \end{aligned}$$

Defining $\theta := f - \frac{1}{2}\chi A^2$, we see that $\theta \in \mathcal{N}(G,\varphi)$ and arrive at the solution in (ii).

That the formulas of (i) and (ii) define solutions of (6) is proved by computation.

The continuity statements follow from the construction and [5, lemma 3.4]. □

3. Application

In this section, using Theorem 2.1 and [11, Theorem 3.3], we solve the functional equation (7), i.e.,

$$f(xy) + g(\varphi(y)x) = h(x)h(y), \quad x, y \in G,$$

by expressing its solutions in terms of characters and additive functions. Solutions of this equation, in the case where $\varphi^2 = id$, can be found, e.g., in [1, 8, 10].

Theorem 3.1. *The triplet $f, g, h : G \to \mathbb{C}$ satisfies the functional equation (7) if and only if it has one of the following forms.*

(a) There exists $\theta \in \mathcal{N}(G,\varphi)$ such that $f = \theta$, $g = -\theta$ and $h = 0$.

(b) There exist a character χ of G with $\chi \circ \varphi \neq \chi$ and $\chi \circ \varphi^2 = \chi$, constants $\alpha, \beta \in \mathbb{C}$, and a function $\theta \in \mathcal{N}(G,\varphi)$ such that

$$\begin{aligned} f &= \alpha^2\chi + \beta^2\chi \circ \varphi + \theta, \\ g &= \alpha\beta(\chi + \chi \circ \varphi) - \theta, \\ h &= \alpha\chi + \beta\chi \circ \varphi. \end{aligned}$$

In this case $h \neq 0$.

(c) There exist a character χ of G with $\chi \circ \varphi = \chi$, a constant $\alpha \in \mathbb{C}$, an additive function $A : G \to \mathbb{C}$ with $A \circ \varphi = -A$, and a function $\theta \in \mathcal{N}(G,\varphi)$ such that

$$\begin{aligned} f &= (\alpha A + \frac{1}{4}A^2)\chi + \theta, \\ g &= (\alpha^2 - \frac{1}{4}A^2)\chi - \theta, \\ h &= (\alpha + A)\chi. \end{aligned}$$

In this case $h \neq 0$.

Moreover, if G is a topological group, φ is continuous and $f, h \in C(G)$, then $g, \chi, \chi \circ \varphi, A, \theta \in C(G)$.

Proof. Let (f, g, h) be a solution of (7). First, suppose that $h = 0$. Then $g = -f$ (to get this replace y by e in (7)) and hence Eq. (7) becomes

$$f(xy) - f(\varphi(y)x) = 0.$$

So $f \in \mathcal{N}(G, \varphi)$. This is case (a) of our statement.

From now, on we assume that $h \neq 0$. Then we have only the following three cases:

Case 1: Suppose $g = 0$. Then $f \neq 0$ (because $h \neq 0$) and hence Eq. (7) becomes

$$f(xy) = h(x)h(y), \quad x, y \in G. \tag{11}$$

Letting here $x = e$, we get $f = h(e)h$. So $h(e) \neq 0$ (because $f \neq 0$). Substituting f in (11), we see that $\frac{h}{h(e)}$ is a character of G. So we are case (b) (with $\alpha = h(e)$, $\beta = 0$ and $\theta = 0$) or case (c) (with $\alpha = h(e)$, $A = 0$ and $\theta = \alpha^2\chi$).

Case 2: Suppose $f = 0$. Then $g \neq 0$ and Eq. (7) becomes

$$g(\varphi(y)x) = h(x)h(y), \quad x, y \in G. \tag{12}$$

From this equation, we see that $g \circ \varphi = g$ and $h \circ \varphi = h$. Therefore, the pair (g, h) satisfies (11) and arrive at the solution in case (c) of our statement with $\alpha = h(e)$ and $A = \theta = 0$.

Case 3: We now suppose that $f \neq 0$ and $g \neq 0$. Putting $y = e$ in (7) we obtain

$$f + g = h(e)h. \tag{13}$$

So Eq. (7) becomes

$$f(xy) - f(\varphi(y)x) = h(x)h(y) - h(e)h(\varphi(y)x), \quad x, y \in G. \tag{14}$$

If $h(e) = 0$, then $g = -f$ and the pair $(2f, h)$ satisfies (6). Since $h \neq 0$, the pair $(2f, h)$ has the form (i) or (ii) of Theorem 2.1. So we are in case (b) (with $\alpha = -\beta = c$) or case (c) (with $\alpha = 0$) of our statement. From now, on we assume that $h(e) \neq 0$. In (7) we first replace y by e, then x by e. We obtain

$$g \circ \varphi = g. \tag{15}$$

Replacing x by $\varphi(y)$ and x by $\varphi(y)$ in (7) and using (15), we get

$$f(\varphi(y)x) + g(xy) = h(x)h(\varphi(y)), \quad x, y \in G.$$

Summing this last equation and (7) and using (13), we get

$$H(xy) + H(\varphi(y)x) = 2H(x)(\frac{H + H \circ \varphi}{2})(y), \quad x, y \in G, \tag{16}$$

where $H = \frac{h}{h(e)}$. So $(H, \frac{H + H\circ\varphi}{2})$ satisfies Wilson functional equation with an endomorphism, i.e. the functional equation $k(xy) + k(\varphi(y)x) = 2k(x)s(y)$, which was solved on groups in [11]. According to [11, Theorem 3.3] there are only the following three possibilities:

(i) There exists a character χ of G such that $H=\gamma\chi$ for some $\gamma\in\mathbb{C}\backslash\{0\}$. Since $H(e)=1$ we have $\gamma=1$, so that $h=h(e)H=h(e)\chi$ and $g=(h(e))^2\chi-f$. Using (14) we obtain

$$f(xy)-f(\varphi(y)x)=(h(e))^2\chi(xy)-(h(e))^2\chi(\varphi(y)x).$$

for all $x,y\in G$. Defining $\theta_1=f-(h(e))^2\chi$, we see that $\theta_1\in\mathcal{N}(G,\varphi)$ and arrive at the solution in case (b) (with $\alpha=h(e)$ and $\beta=0$) or case (c) (with $\alpha=h(e)$, $A=0$ and $\theta=\theta_1+\alpha^2\chi$) of our statement.

(ii) There exists a character χ of G with $\chi\circ\varphi\neq\chi$ and $\chi\circ\varphi^2=\chi$ such that

$$H=\gamma\chi+\delta\chi\circ\varphi \tag{17}$$

for some $\gamma,\delta\in\mathbb{C}\backslash\{0\}$. Since $H(e)=1$ we have $\gamma+\delta=1$. Hence

$$h=h(e)H=h(e)[\gamma\chi+(1-\gamma)\chi\circ\varphi]. \tag{18}$$

Using (14) and the fact that $\chi\circ\varphi^2=\chi$ we obtain

$$\begin{aligned} f(xy)-f(\varphi(y)x) &= (h(e))^2[(\gamma^2\chi+(1-\gamma)^2\chi\circ\varphi)(xy) \\ &\quad -(\gamma^2\chi+(1-\gamma)^2\chi\circ\varphi)(\varphi(y)x)] \end{aligned}$$

Defining $\theta:=f-(h(e))^2[\gamma^2\chi+(1-\gamma)^2\chi\circ\varphi]$, we see that $\theta\in\mathcal{N}(G,\varphi)$ and arrive at the solution in case (b) above with $\alpha=\gamma h(e)$ and $\beta=(1-\gamma)h(e)$.

(iii) There exist a character χ of G with $\chi\circ\varphi=\chi$ and a non-zero additive function $a:G\to\mathbb{C}$ with $a\circ\varphi=-a$ such that

$$H=(\gamma+a)\chi \tag{19}$$

for some $\gamma\in\mathbb{C}$. Since $H(e)=1$ we have $\gamma=1$. Then

$$h=h(e)H=(h(e)+h(e)a)\chi.$$

Since $a\circ\varphi=-a$, we have

$$a(x)a(y)=\frac{1}{4}a^2(xy)-\frac{1}{4}a^2(\varphi(y)x)\quad\text{for all}\quad x,y\in G.$$

Using this last equality and the fact that $\chi\circ\varphi=\chi$ in (14), we obtain

$$\begin{aligned} f(xy)-f(\varphi(y)x) &= (h(e))^2[(1+a(x))(1+a(y)) \\ &\quad -(1+a(\varphi(y)x))]\chi(xy) \\ &= (h(e))^2[(a+\frac{1}{4}a^2)(xy)-(a+\frac{1}{4}a^2)(\varphi(y)x)]. \end{aligned}$$

Defining $\theta:=f-(h(e))^2[a+\frac{1}{4}a^2]\chi$, we see that $\theta\in\mathcal{N}(G,\varphi)$ and arrive at the solution in case (c) above with $\alpha=h(e)$ and $A=h(e)a$.

Conversely, it is a simple calculation to check that in all cases listed above the given functions are solutions of the functional equation (7).

Finally, the continuity statements follow from the construction, equality (13) and [12, Theorem 3.18(d)]. $\square$

In the following corollary, we determine all solutions of an important special case of Eq. (7) on the real line.

Corollary 3.1. *Let* $z_0 \in \mathbb{R}\backslash\{-1,1\}$ *be a fixed element. The only solutions* $f, g, h : \mathbb{R} \to \mathbb{C}$ *of the functional equation*

$$f(x+y) + g(x+z_0y) = h(x)h(y), \quad x, y \in \mathbb{R}, \tag{20}$$

are $f = c$, $g = \alpha^2 - c$ *and* $h = \alpha$ *where* $\alpha, c \in \mathbb{C}$.

Proof. We apply Theorem 3.1 with $G = (\mathbb{R}, +)$ and $\varphi(x) = z_0x$ for all $x \in \mathbb{R}$.

Let χ be a character of $\mathbb{R}$. So $\chi(x) \neq 0$ for all $x \in \mathbb{R}$. If $\chi(z_0^2x) = \chi(x)$ for all $x \in \mathbb{R}$, then $\chi((z_0^2-1)x) = 1$ for all $x \in \mathbb{R}$. But $(z_0^2 - 1)\mathbb{R} = \mathbb{R}$, because $z_0 \neq \pm 1$, so we get that $\chi = 1$.

Let $A : \mathbb{R} \to \mathbb{C}$ be an additive function. If $A(z_0x) = -A(x)$ for all $x \in \mathbb{R}$, then A is a constant. So that $A = 0$.

Let $\theta \in \mathcal{N}(G, \varphi)$. Then $\theta(x + y) = \theta(x + z_0y)$ for all $x, y \in \mathbb{R}$, i.e., $\theta(x) = \theta(x + (z_0 - 1)y)$ for all $x, y \in \mathbb{R}$. Since $\mathbb{R}$ is divisible by $z_0 - 1$, because $z_0 \neq 1$, then θ is a constant.

Finally, from these results we infer that the only solutions of Eq. (20) are those stated in corollary 3.1. □

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