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# Rough Approximations in Topological Vector Spaces

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**Abstract:** In this article, we propose the lower and upper approximations in a topological vector space. Also we prove some algebraic and topological results of these approximations and give some implications of algebraic and set operations on these approximations.

**Key words:** Rough approximations, lower approximation, upper approximation, Vector Space, Topological Vector Space

## 1. Introduction

The beginning of rough set theory was in 1982 by Z. Pawlak [1]. He has proposed that this theory is based on the classification of the universe $U$ according to an equivalence relation $R$ that defined on $U$, a pair $(U, R)$ is called an approximation space. Each equivalence class in that space is called an elementary set and the finite union of elementary sets is called a composed set. Any subset $X$ of a universe described by two composed sets, lower and upper approximations of $X$. A lower approximation of $X$ is the union of all elementary sets that contained in $X$, and an upper approximation of $X$ is the union of all elementary sets that intersect with $X$. Recently some researchers have applied the rough approximations in groups [2, 3] and topological groups [4]. In 2009 Mingfen, Xiangyun and Cungen applied the concepts of lower and upper approximations in vector subspaces and presented some results of them.

In this work, we are going to apply some results in [4] on the concept of rough approximations in a vector space that proposed in [5], and then apply the concept of lower and upper approximations in a topological vector space. The organization of the paper as the following way: In Section 2 the elementary of rough set theory will be given. In Section 3, we review the rough approximations in the vector space and some theorems that are proved in [5], we further introduce and prove some other results of them. In Section 4 we define rough approximations in a topological vector space, and rewrite the some results which have been mentioned in [4, 5] into topological vector spaces.

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## 2. Preliminaries

In this section, we shall illustrate some basic notions and results in [1, 6] will be needed in this paper. Throughout this article, $V$ denotes a vector space unless stated otherwise.

**Definition 2.1.** A pair $(V, \mathcal{T})$ of a vector space $V$ over a field $\mathbb{F}$ endowed with a topology $\mathcal{T}$ on $V$ is called a **topological vector space** over $\mathbb{F}$ if these two axioms hold:


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1. The map $(x, y) \to x + y$ from $V \times V$ into $V$ is continuous; and
2. The map $(\alpha, x) \to \alpha x$ from $\mathbb{F} \times V$ into $V$ is continuous.

Whereas the topology on the field $\mathbb{F}$ is the usual topology on $\mathbb{R}$ or $\mathbb{C}$.

**Definition 2.2.** A pair $(U, R)$ of a nonempty set $U$ and an equivalence relation $R$ is called an approximation space.

**Definition 2.3.** If $(U, R)$ is an approximation space, then for each $X \subseteq U$, we define two sets called lower and upper approximations of $X$, denoted by $\underline{App}(X)$ and $\overline{App}(X)$ respectively, as follows:

1. $\underline{App}(X) = \{t : [t]_R \subseteq X\}$,
2. $\overline{App}(X) = \{t : [t]_R \cap X \neq \emptyset\}$,

where $[t]_R$ denotes the equivalence class of $t$ with respect to the equivalence relation $R$.

For each subsets $X, Y$ in $(U, R)$ we get the followings:

1. $\underline{App}(Y) \subseteq Y \subseteq \overline{App}(Y)$,
2. $\overline{App}(U) = \underline{App}(U) = U$,
3. $\overline{App}(\phi) = \underline{App}(\phi) = \phi$,
4. $\overline{App}(Y \cup X) = \overline{App}(Y) \cup \overline{App}(X)$,
5. $\underline{App}(Y \cap X) = \underline{App}(Y) \cap \underline{App}(X)$,
6. $\overline{App}(Y \cap X) \subseteq \overline{App}(Y) \cap \overline{App}(X)$,
7. $\underline{App}(Y) \cup \underline{App}(X) \subseteq \underline{App}(Y \cup X)$,
8. $\overline{App}(Y) \subseteq \overline{App}(X)$ and $\underline{App}(Y) \subseteq \underline{App}(X)$ whenever $Y \subseteq X$.

**Definition 2.4.** A composed set $C$ is a subset of an approximation space, such that,

$$\underline{App}(C) = \overline{App}(C) = C.$$

## 3. Rough Approximations in a Vector space

In this section, the concept of rough approximations of a set on a vector space and some theorems which proposed by Mingfen, Xiangyun and Cungen in [5] will be represented. Additionally we are going to prove some other results about it.

**Definition 3.1.** [5] If $V$ is a vector space and $K$ is a subspace of $V$, then for a nonempty subset $X$ of $V$, the following sets

$$\begin{aligned} \underline{K}(X) &= \{x \in V : x + K \subseteq X\} \\ &= \textstyle\bigcup_{x \in V} \{x + K : x + K \subseteq X\}, \text{ and} \\ \overline{K}(X) &= \{x \in V : (x + K) \cap X \neq \emptyset\} \\ &= \textstyle\bigcup_{x \in V} \{x + K : (x + K) \cap X \neq \emptyset\} \end{aligned}$$

are called **lower** and **upper approximations** of the set $X$ with respect to the subspace $K$, respectively.

**Definition 3.2.** Let $V$ be a vector space, $K$ be a subspace of $V$ and $X$ be a nonempty subset of $V$. Then $X$ is called an **upper(lower) rough subspace** if $\overline{K}(X)\big(\underline{K}(X)\big)$ is also a subspace of $V$.

**Theorem 3.1.** *[5] Let $K$ be a subspace of a vector space $V$, and let $X$ and $Y$ be any nonempty subsets of $V$. Then the followings are true:*

1. $\underline{K}(X) \subseteq X \subseteq \overline{K}(X)$,
2. $\overline{K}(X \cup Y) = \overline{K}(X) \cup \overline{K}(Y)$,
3. $\underline{K}(X \cap Y) = \underline{K}(X) \cap \underline{K}(Y)$,
4. $\underline{K}(X) \cup \underline{K}(Y) \subseteq \underline{K}(X \cup Y)$,
5. $\overline{K}(X \cap Y) \subseteq \overline{K}(X) \cap \overline{K}(Y)$,
6. *If* $X \subseteq Y$, *then* $\underline{K}(X) \subseteq \underline{K}(Y)$ *and* $\overline{K}(X) \subseteq \overline{K}(Y)$.

**Theorem 3.2.** *Let $K$ be a subspace of a vector space $V$. Let $X$ be any subset of $V$. Then the following properties are satisfied:*

1. $\underline{K}(\emptyset) = \overline{K}(\emptyset) = \emptyset$, *and* $\underline{K}(V) = \overline{K}(V) = V$,
2. $\underline{K}(V \backslash X) = V \backslash \overline{K}(X)$,
3. $V \backslash \underline{K}(X) \subseteq \overline{K}(V \backslash X)$.

*Proof.*

1. There does not exist $x \in V$ such that $x + \emptyset \subseteq \emptyset$, so $\underline{K}(\emptyset) = \emptyset$ and for each $x \in V$, $(x + \emptyset) = \emptyset$. Thus, $\overline{K}(\emptyset) = \emptyset$. It is obvious from the Definition 3.1, $\underline{K}(V) = \overline{K}(V) = V$ is true.
2. $x \in \underline{K}(V \backslash X)$ if and only if $(x + K) \subseteq V \backslash X$ if and only if $(x + K) \cap X = \emptyset$ if and only if $x \notin \overline{K}(X)$ if and only if $x \in V \backslash \overline{K}(X)$. Thus, $\underline{K}(V \backslash X) = V \backslash \overline{K}(X)$.
3. We claim that: $V \backslash \overline{K}(V \backslash X) \subseteq \underline{K}(X)$. Let $x \in V \backslash \overline{K}(V \backslash X)$. Then $x \notin \overline{K}(V \backslash X)$. So $(x + K) \cap (V \backslash X) = \emptyset$ and $(x + K) \subseteq X$. Thus, $x \in \underline{K}(X)$ and the desired holds.

□

**Theorem 3.3.** *If $K$ and $X$ are subspaces of a vector space $V$. Then*

$$\underline{K}(X) = \begin{cases} \emptyset & \textit{whenever } K \nsubseteq X \\ X & \textit{whenever } K \subseteq X. \end{cases}$$

*Proof.* First, we need to prove that $\underline{K}(X) = \emptyset$ if $K \nsubseteq X$. Suppose to the contrary that there is an element $x \in V$ such that $(x + K) \subseteq X$, then $x = x + 0 \in X$ where $0$ is the vector space identity in $K$. Since $X$ is a subspace, then $-x \in X$. Then we get, $K \subseteq X$ because $-x + (x + K) \subseteq X + X \subseteq X$. Which is a contradiction.

Hence, for each $x \in V$, $(x+K) \nsubseteq X$ if $K \nsubseteq X$, that is, $\underline{K}(X) = \emptyset$.
Second, suppose that $K \subseteq X$. Since $X$ is a subspace, then for each $x \in X$, $x + K \subseteq X + X \subseteq X$. Thus, $x \in \underline{K}(X)$ and it follows that $X \subseteq \underline{K}(X)$. This together with the first property in Theorem 3.1, we have $\underline{K}(X) = X$. □

**Theorem 3.4.** *[5] Let $K$ be a subspace of a vector space $V$ and $\emptyset \neq X \subseteq V$. Then*

$$\overline{K}(X) = X + K.$$

**Corollary 3.1.** *Let $K$ and $X$ be subspaces of a vector space V. If $K \subseteq X$, then $X$ is a composed set.*

*Proof.* By Theorem 3.3 we already have $\underline{K}(X) = X$. Since $K \subseteq X$, then by Theorem 3.4 $\overline{K}(X) = X$. Thus, $\underline{K}(X) = \overline{K}(X) = X.$ □

**Theorem 3.5.** *Let $K_1$, $K_2$, and $X$ be subspaces of a vector space V. Then the followings hold.*

1. $\overline{K_1}(X) + \overline{K_2}(X) = \overline{K_1 + K_2}(X)$.
2. $\underline{K_1}(X) + \underline{K_2}(X) = \underline{K_1 + K_2}(X)$.

*Proof.* Since $X$ is a subspace of $V$, then $X + X = X$.

1. By Theorem 3.4, we get
$$\begin{aligned} \overline{K_1}(X) + \overline{K_2}(X) &= K_1 + X + K_2 + X \\ &= K_1 + K_2 + (X + X) \\ &= K_1 + K_2 + X \\ &= \overline{K_1 + K_2}(X). \end{aligned}$$

2. If $x \in \underline{K_1}(X) + \underline{K_2}(X)$, then there are $\alpha \in \underline{K_1}(X)$ and $\beta \in \underline{K_2}(X)$ such that $x = \alpha + \beta$. By Definition 3.1, we have $\alpha + K_1 \subseteq X$ and $\beta + K_2 \subseteq X$. Hence
$$\begin{aligned} x + (K_1 + K_2) &= (\alpha + \beta) + (K_1 + K_2) \\ &= (\alpha + K_1) + (\beta + K_2) \\ &\subseteq X + X \\ &= X, \end{aligned}$$
and so $x \in \underline{K_1 + K_2}(X)$. Therefore
$$\underline{K_1}(X) + \underline{K_2}(X) \subseteq \underline{K_1 + K_2}(X). \tag{1}$$
Conversely, if $\underline{K_1}(X)$ and $\underline{K_2}(X)$ are nonempty, then by Theorem 3.3,
$$\underline{K_1}(X) = X = \underline{K_2}(X),$$
and according to the Property 1 of properties of an approximation space, $\underline{K_1 + K_2}(X) \subseteq X$. So,
$$\underline{K_1 + K_2}(X) \subseteq X = X + X = \underline{K_1}(X) + \underline{K_2}(X). \tag{2}$$
Hence from (1) and (2) the equality holds.

□

## 4. Rough Approximations in a Topological Vector Space

In this section, the concept of lower and upper approximations are applied in a topological vector space. Additionally, some results of these notions will be given.

**Definition 4.1.** Let $K$ be a subspace of a topological vector space $V$ and $X$ be a nonempty subset of $V$. Then the following sets

$$\begin{aligned}\underline{K}(X) &= \{x \in V : x + K \subseteq X\}\\ &= \bigcup_{x\in V}\{x + K : x + K \subseteq X\}\end{aligned}$$

is called **lower approximation** of $X$ with respcet to $K$, and

$$\begin{aligned}\overline{K}(X) &= \{x \in V : (x + K) \cap X \neq \emptyset\}\\ &= \bigcup_{x\in V}\{x + K : (x + K) \cap X \neq \emptyset\}\end{aligned}$$

is called **upper approximation** of $X$ with respect to $K$.

The properties of rough approximations in Theorems 3.1 and 3.2 also apply on rough approximations in Definition 4.1.

**Theorem 4.1.** *Let $K$ be a subspace of a topological vector space $V$, and $X, Y$ be any subsets of $V$. Then the following properties are satisfied:*

1. $\underline{K}(X) \subseteq X \subseteq \overline{K}(X)$,
2. $\underline{K}(\emptyset) = \overline{K}(\emptyset) = \emptyset$, *and* $\underline{K}(V) = \overline{K}(V) = V$,
3. $\overline{K}(X \cup Y) = \overline{K}(X) \cup \overline{K}(Y)$,
4. $\underline{K}(X \cap Y) = \underline{K}(X) \cap \underline{K}(Y)$,
5. $\underline{K}(X) \cup \underline{K}(Y) \subseteq \underline{K}(X \cup Y)$,
6. $\overline{K}(X \cap Y) \subseteq \overline{K}(X) \cap \overline{K}(Y)$,
7. *If* $X \subset Y$, *then* $\underline{K}(X) \subseteq \underline{K}(Y)$ *and* $\overline{K}(X) \subseteq \overline{K}(Y)$,
8. $\underline{K}(V \backslash X) = V \backslash \overline{K}(X)$,
9. $V \backslash \underline{K}(X) \subseteq \overline{K}(V \backslash X)$.

**Theorem 4.2.** *Let $K$ be a subspace of a topological vector space $V$ and $X$ be any nonempty subset of $V$. If $K$ is open, then $\underline{K}(X)$ and $\overline{K}(X)$ are also open.*

*Proof.* Let $K$ be an open subset of a topological vector space $V$. Then $x + K$ is also open since $K$ and $x + K$ are homeomorphic, because the addition is homeomorphism. So, by the Definition 4.1 $\underline{K}(X)$ and $\overline{K}(X)$ are the union of open sets, then they are also open. □

**Theorem 4.3.** *Let $K$ be a subspace of a finite topological vector space $V$ and $X$ be any nonempty subset of $V$. If $K$ is closed, then $\underline{K}(X)$ and $\overline{K}(X)$ are also closed.*

*Proof.* Let $K$ be a closed subset of a topological vector space $V$. Then $x+K$ is also closed since $K$ and $x+K$ are homeomorphic, because the addition is homeomorphism. So, by the Definition 4.1 $\underline{K}(X)$ and $\overline{K}(X)$ are the finite union of closed sets, then they are also closed. □

**Theorem 4.4.** *If $K$ and $X$ are subspaces of a topological vector space $V$. Then*

$$\underline{K}(X) = \begin{cases} \emptyset & \textit{whenever } K \nsubseteq X \\ X & \textit{whenever } K \subseteq X. \end{cases}$$

*Proof.* The proof exactly like the proof of Theorem 3.3. □

**Theorem 4.5.** *Let $K$ be a subspace of a topological vector space $V$ and $\emptyset \neq X \subseteq V$. Then*

$$\overline{K}(X) = X + K.$$

*Proof.* Let $x$ be an arbitrary element of $\overline{K}(X)$. Then $(x+K)\cap X \neq \emptyset$ which implies there is an element $y \in V$ such that $y \in X$ and $y \in (x+K)$. That is, $y = x+z$ where $z \in K$. Thus, $x = y + (-z) \in (y+K)$ since $K$ is a subspace of $V$. So, $x \in X + K$ because $y \in X$, and hence $\overline{K}(X) \subseteq X + K$.
On the other direction, let $x$ be any element in $X + K$. Then there are elements $a \in X$ and $z \in K$ such that $x = a + z$. Then $a = x - z$ and $a \in (x+K)$ since $K$ is a subspace. So, $a \in (x+K)\cap X$ which implies that $(x+K)\cap X \neq \emptyset$ and $a \in \overline{K}(X)$. Then, $X + K \subseteq \overline{K}(X)$ and therefore, $\overline{K}(X) = X + K$. □

**Corollary 4.1.** *Let $K$ and $X$ be subspaces of a topological vector space $V$. If $K \subseteq X$, then $X$ is a composed set.*

*Proof.* The proof exactly like the proof of Corollary 3.1. □

**Theorem 4.6.** *If $K_1$ and $K_2$ are subspaces of a topological vector space $V$, and $X, Y \subseteq V$. Then*

1. $\overline{K_1}(X+Y) = \overline{K_1}(X) + \overline{K_1}(Y)$.

2. $\underline{K_1}(X) + \underline{K_1}(Y) \subseteq \underline{K_1}(X+Y)$.

3. $\overline{(K_1+K_2)}(X+Y) = \overline{K_1}(X) + \overline{K_2}(Y)$.

4. $\underline{K_1}(X) + \underline{K_2}(Y) \subseteq \underline{(K_1+K_2)}(X+Y)$.

*Proof.* The proof is similar to Theorems 4.5 and 4.7 of [5]. □

**Example 4.1.** *Let the vector space $V$ be the Euclidean plane $\mathbb{R}^2$ over the field $\mathbb{R}$ with the usual topology $\mathcal{U}$, and*

$$\begin{aligned} X &= \{\langle x, 0\rangle : \ x \in \mathbb{R}\}, \\ Y &= \{\langle 0, y\rangle : \ y \in \mathbb{R}\}, \\ K_1 &= \{\langle x, y\rangle : y = x; \ x \in \mathbb{R}\}, \\ K_2 &= \{\langle x, y\rangle : y = -x; \ x \in \mathbb{R}\}. \end{aligned}$$

*So, $K_1, K_2, X$ and $Y$ are all subspaces of $V$. By Theorem 4.4 we have $\underline{K_1}(X) = \emptyset = \underline{K_1}(Y)$, and by Theorem 4.5 the upper approximations of $X$ and $Y$ are*

$$\overline{K_1}(X) = K_1 + X = \{\langle x + \alpha, \alpha \rangle : \alpha, x \in \mathbb{R}\}, \textit{and}$$
$$\overline{K_1}(Y) = K_1 + Y = \{\langle \alpha, y + \alpha \rangle : \alpha, y \in \mathbb{R}\},$$

*respectively. Note that $x-$ axis is not open in $\mathbb{R}^2$ with the product of usual topology because any point on the $x-$ axis has no open neighborhood which is contained in $X$, but it is already closed since its complement is open. Note that $X + Y = \{\langle x, 0 \rangle + \langle 0, y \rangle : x, y \in \mathbb{R}\} = \mathbb{R}^2 = V$. Now we review the validity of Theorem 4.6.*

1. $\overline{K_1}(X+Y) = X+Y+K_1$, $\overline{K_1}(X) = X+K_1$, $\overline{K_1}(Y) = Y+K_1$ *and since* $K_1$ *is a subspace* $\overline{K_1}(X)+\overline{K_1}(Y) = X + K_1 + Y + K_1 = X + Y + (K_1 + K_1) = X + Y + K_1 = \overline{K_1}(X + Y)$.
2. $\underline{K_1}(X) + \underline{K_1}(Y) = \emptyset + \emptyset = \emptyset$ *and* $\underline{K_1}(X + Y) = \underline{K_1}(V) = V$, *then* $\underline{K_1}(X) + \underline{K_1}(Y) \subseteq \underline{K_1}(X + Y)$.
3. $\overline{(K_1 + K_2)}(X + Y) = K_1 + K_2 + (X + Y) = K_1 + K_2 + V = V$ *and* $\overline{K_1}(X) + \overline{K_2}(Y) = X + K_1 + Y + K_2 = K_1 + K_2 + (X + Y) = K_1 + K_2 + V = V$, *so* $\overline{(K_1 + K_2)}(X + Y) = \overline{K_1}(X) + \overline{K_2}(Y)$.
4. $\underline{(K_1 + K_2)}(X + Y) = V$ *since* $K_1 + K_2 \subseteq V$ *by Theorem 4.4, and* $\underline{K_1}(X) + \underline{K_2}(Y) = \emptyset + \emptyset = \emptyset$. *So,* $\underline{K_1}(X) + \underline{K_2}(Y) \subseteq \underline{(K_1 + K_2)}(X + Y)$.

2 *and* 4 *in this example show the inclusion in* 2 *and* 4 *of Theorem 4.6 may be not equal in general.*

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