Asia Mathematika
Volume: 3 Issue: 2 , (2019) Pages: 1 – 18
Available online at **www.asiamath.org**

Study of the group theory in neutrosophic soft sense

Tuhin Bera[1*] and Nirmal Kumar Mahapatra[2]
[1]Department of Mathematics, Boror S. S. High School, Bagnan, Howrah-711312, WB, India.
[2]Department of Mathematics, Panskura Banamali College, Panskura RS-721152, WB, India.

Received: 10 Jan 2019 • **Accepted:** 31 May 2019 • **Published Online:** 01 Aug 2019

Abstract: The purpose of this paper is to cultivate the group theory by means of neutrosophic soft sense in a different way. The concepts of neutrosophic soft coset, neutrosophic normal soft group, neutrosophic soft quotient group, direct product of neutrosophic soft groups and simple neutrosophic soft group have been presented in a new approach. These are illustrated by suitable examples. Their structural characteristics are investigated here in the parlance of group theory in classical sense. Two kinds of composition namely binary composition '$\circ$' between the elements of a classical group and neutrosophic soft composition / neutrosophic soft product 'o' between the neutrosophic soft elements of neutrosophic soft groups are used to practice here. Following the classical group theory, the concepts have been developed by using the neutrosophic soft composition directly.

Key words: Neutrosophic soft coset, Neutrosophic normal soft group, Neutrosophic soft quotient group, Direct product, Simple neutrosophic soft groups.

1. Introduction

Molodtsov [13] brought an opportunity to handle the uncertainty more precisely by introducing 'Soft set theory'. Researchers in several real fields deal daily with the complexities of modeling uncertain data. There are different useful tools like probability theory, theory of fuzzy set [19], intuitionistic fuzzy set [3] etc to describe uncertainty. But the parametrization inadequacy makes all these efforts unfruitful. In that ground, soft set theory is remarkable because of it's parametrization adequacy. Several authors [1, 2, 14, 15, 17, 18] extended the different algebraic structures over fuzzy set, intuitionistic fuzzy set and soft set.

A more generalisation of classical sets, fuzzy set, intuitionistic fuzzy set is 'neutrosophic set' (NS) revealed by Smarandache [16]. It is recently being practiced in development of various mathematical structures and decision making. The decision makers can get an opportunity to include their hesitation in decision making by this theory. Intuitionistic fuzzy set theory can not meet that point. Another advantage of NS theory over intuitionistic fuzzy set is that the characters representing an object are independent and appear explicitly. The combination of NS and soft set was given first by Maji [12] and thus the notion of 'neutrosophic soft set' (Nss) was brought to light. This concept has been practiced by several researchers [4–11] to develop different tracks of mathematics.

This paper helps to investigate the characteristics related to neutrosophic soft group in a new direction. After given some preliminary useful definitions in Section 2, the study is moved to Section 3 to state the main results. Here, the concept of neutrosophic soft coset, neutrosophic normal soft group, neutrosophic soft quotient group, direct product of neutrosophic soft groups and simple neutrosophic soft group are introduced in a new

*Correspondence: tuhin78bera@gmail.com

direction following the sense of classical group theory. Finally, the Section 4 deals the present work at a glance.

2. Preliminaries

We recall some necessary definitions and results to make out the main thought.

Definition 2.1. [4] A continuous t- norm $*$ and t- conorm $\diamond$ are two continuous binary operations assigning $[0,1]\times[0,1]\to[0,1]$ and obey the under stated principles :
(i) $*$ and $\diamond$ are both commutative and associative.
(ii) $u*1=1*u=u$ and $u\diamond 0=0\diamond u=u,\ \forall u\in[0,1]$.
(iii) $u*v\le p*q$ and $u\diamond v\le p\diamond q$ if $u\le p,\ v\le q$ with $u,v,p,q\in[0,1]$.
$u*v=uv, u*v=\min\{u,v\}, u*v=\max\{u+v-1,0\}$ are most useful t-norms and
$u\diamond v=u+v-uv, u\diamond v=\max\{u,v\}, u\diamond v=\min\{u+v,1\}$ are most useful t-conorms.

Definition 2.2. [13] A soft set on an initial universe X is presented by a pair (M,D) where $D\subseteq E$, the parametric set and M maps $D\to\wp(X)$, the power set of X.

Definition 2.3. [16] An NS Q on an initial universe X is presented by three characterisations namely true value T_Q, indeterminant value I_Q and false value F_Q so that $T_Q,I_Q,F_Q:X\to]^-0,1^+[$. Thus Q can be designed as : $\{<u,(T_Q(u),I_Q(u),F_Q(u))>: u\in X\}$ with $^-0\le\sup T_Q(u)+\sup I_Q(u)+\sup F_Q(u)\le 3^+$. Here $1^+=1+\delta$, where 1 is standard part and δ is non-standard part. Similarly $^-0=0-\delta$. The non-standard set $]^-0,1^+[$ is basically practiced in philosophical ground and because of the difficulty to adopt it in real field, the standard subset of $]^-0,1^+[$ i.e., [0,1] is applicable in real neutrosophic environment.

Definition 2.4. [12] An Nss on an initial universe X is presented by a pair (N,B) where $B\subseteq E$, the parametric set and N maps $B\to NS(X)$, the set of all NSs of X.

Deli and Broumi [11] proposed this notion in a new look.

Definition 2.5. [11] An Nss Q on (X,E), X being the universe set and E being the parametric set, is presented by an ordered pair $(e,f_Q(e)), e\in E$ where f_Q maps $E\to NS(X)$, the set of all NSs on X and is given by $f_Q(e)=\{<u,(T_{f_Q(e)}(u),I_{f_Q(e)}(u),F_{f_Q(e)}(u))>: u\in X\}$ with $T_{f_Q(e)}(u),I_{f_Q(e)}(u),F_{f_Q(e)}(u)\in[0,1]$ and $0\le T_{f_Q(e)}(u)+I_{f_Q(e)}(u)+F_{f_Q(e)}(u)\le 3$.

Definition 2.6. [5] Consider two Nss P and Q on the common universe U via parametric set E. Then,
1. P is called neutrosophic soft subset of Q, denoted as $P\subseteq Q$, when

$$T_{f_P(e)}(u)\le T_{f_Q(e)}(u), I_{f_P(e)}(u)\ge I_{f_Q(e)}(u), F_{f_P(e)}(u)\ge F_{f_Q(e)}(u), \forall e\in E, u\in X.$$

2. the 'AND' operation $(P\wedge Q)$ is also an Nss and is defined by :
$M=\{[(e,e'),\{<u,T_{f_M(e,e')}(u),I_{f_M(e,e')}(u),F_{f_M(e,e')}(u)>: u\in X\}]:(e,e')\in E\times E\}$
where $T_{f_M(e,e')}(u)=T_{f_P(e)}(u)*T_{f_Q(e')}(u), I_{f_M(e,e')}(u)=I_{f_P(e)}(x)\diamond I_{f_Q(e')}(u)$ and
$F_{f_M(e,e')}(u)=F_{f_P(e)}(u)\diamond F_{f_Q(e')}(u)$.

3. the 'OR' operation $(P\vee Q)$ is also an Nss and is defined by :
$K=\{[(e,e'),\{<u,T_{f_K(e,e')}(u),I_{f_K(e,e')}(u),F_{f_K(e,e')}(u)>: u\in X\}]:(e,e')\in E\times E\}$
where $T_{f_K(e,e')}(u)=T_{f_P(e)}(u)\diamond T_{f_Q(e')}(u), I_{f_K(e,e')}(u)=I_{f_P(e)}(u)* I_{f_Q(e')}(u)$ and
$F_{f_K(e,e')}(u)=F_{f_P(e)}(u)* F_{f_Q(e')}(u)$.

Definition 2.7. [6] An Nss function (ψ, ξ) is presented by $(X, E) \longrightarrow (Y, E)$ where $\psi : X \to Y$ and $\xi : E \to E$. Define two Nss P on (X, E) and Q on (Y, E). Then,

(1) the image of P under (ψ, ξ) is an Nss $(\psi, \xi)(P)$ on (Y, E) and it is defined as : $(\psi, \xi)(P) = \{< \xi(a), f_{\psi(P)}(\xi(a)) >: a \in E\}$ where $\forall b \in \xi(E), \forall v \in Y$,

$$T_{f_{\psi(P)}(b)}(v) = \begin{cases} \max_{\psi(u)=v} \max_{\xi(a)=b} [T_{f_P(a)}(u)], & \text{for } u \in \psi^{-1}(v) \\ 0 \quad , & \text{otherwise.} \end{cases}$$

$$I_{f_{\psi(P)}(b)}(v) = \begin{cases} \min_{\psi(u)=v} \min_{\xi(a)=b} [I_{f_P(a)}(u)], & \text{for } u \in \psi^{-1}(v) \\ 1 \quad , & \text{otherwise.} \end{cases}$$

$$F_{f_{\psi(P)}(b)}(v) = \begin{cases} \min_{\psi(u)=v} \min_{\xi(a)=b} [F_{f_P(a)}(u)], & \text{for } u \in \psi^{-1}(v) \\ 1 \quad , & \text{otherwise.} \end{cases}$$

(2) the pre-image of Q under (ψ, ξ), is an Nss $(\psi, \xi)^{-1}(Q)$ on (X, E) and it is defined as, $\forall a \in \xi^{-1}(E), \forall u \in X$,

$$T_{f_{\psi^{-1}(Q)}(a)}(u) = T_{f_Q[\xi(a)]}(\psi(u)), I_{f_{\psi^{-1}(Q)}(a)}(u) = I_{f_Q[\xi(a)]}(\psi(u)), F_{f_{\psi^{-1}(Q)}(a)}(u) = F_{f_Q[\xi(a)]}(\psi(u)).$$

(ψ, ξ) is injective (surjective) when ψ and ξ both are injective (surjective).

Definition 2.8. [4] The neutrosophic soft product of Nss P and Q defined on a groupoid G is denoted by PoQ and it is also an Nss S defined as, for $(a, b) \in E \times E$ and $u \in G$,

$$T_{f_{S(a,b)}}(u) = \begin{cases} \max_{u=xv}[T_{f_P(a)}(x) * T_{f_Q(b)}(v)] \\ 0 \quad \text{if } u \text{ can not be put as } u = xv. \end{cases}$$

$$I_{f_{S(a,b)}}(u) = \begin{cases} \min_{u=xv}[I_{f_P(a)}(x) \diamond I_{f_Q(b)}(v)] \\ 1 \quad \text{if } u \text{ can not be put as } u = xv. \end{cases}$$

$$F_{f_{S(a,b)}}(u) = \begin{cases} \min_{u=xv}[F_{f_P(a)}(x) \diamond F_{f_Q(b)}(v)] \\ 1 \quad \text{if } u \text{ can not be put as } u = xv. \end{cases}$$

Definition 2.9. [8] **1.** The null Nss on (X, E) is denoted by ϕ_X and is defined by $(f_{\phi_X}(e))(u) = (0, 1, 1), \forall e \in E, \forall u \in X$.

2. The absolute Nss on (X, E) is denoted by 1_X and is defined by $(f_{1_X}(e))(u) = (1, 0, 0), \forall e \in E, \forall u \in X$.

Clearly, $\phi_X^c = 1_X$ and $1_X^c = \phi_X$.

Definition 2.10. [8] 1. An NS $(e, f_Q(e))$, $e \in E$ in an Nss Q over (X, E) is called a neutrosophic soft point denoted by e_Q, if $f_Q(e) \notin \phi_X$ and $f_Q(e') \in \phi_X$ $\forall e' \in E - \{e\}$.

2. The complement of e_Q is also a neutrosophic soft point e_Q^c such that $f_Q^c(e) = (f_Q(e))^c$ hold.

3. A neutrosophic soft point $e_Q \in P$, an Nss if $f_Q(e) \leq f_P(e)$ for $e \in E$.

Definition 2.11. [5] An NS Q defined on a crisp group $(G, \circ)$ is called a neutrosophic subgroup of $(G, \circ)$ with respect to the following sets of condition.

$$(i) \begin{cases} T_Q(u \circ v) \geq T_Q(u) * T_Q(v) \\ I_Q(u \circ v) \leq I_Q(u) \diamond I_Q(v) \\ F_Q(u \circ v) \leq F_Q(u) \diamond F_Q(v), \forall u, v \in G. \end{cases} \qquad (ii) \begin{cases} T_Q(u^{-1}) \geq T_Q(u) \\ I_Q(u^{-1}) \leq I_Q(u) \\ F_Q(u^{-1}) \leq F_Q(u), \forall u \in G. \end{cases}$$

An Nss Q on $((G, \circ), E)$ will be a neutrosophic soft group (NSG) if $f_Q(e)$ is a neutrosophic subgroup of $(G, \circ)$, $\forall e \in E$.

Over $((G, \circ), E)$, an NSG P is called a neutrosophic soft subgroup of another NSG Q if $P \subseteq Q$.

3. Main Result

Earlier, we have defined the left and right neutrosophic soft coset (NSC) of an Nss P over a classical group G in the paper [6]. Here we shall extend this concept for a neutrosophic soft subgroup M of an NSG P.
Through out the study, unless otherwise stated we shall treat G as a classical group and E as a parametric set.

Definition 3.1. Over $((G,\circ),E)$, define two NSGs M,P with $M\subseteq P$ and $e_1,e_2\in E$. Then for a fixed but arbitrary neutrosophic soft element $f_P(e_1)\in P$, the left NSC of M in P is :

$$\begin{aligned} f_P(e_1)oM &= \{f_P(e_1)of_M(e_2): f_M(e_2)\in M\} \\ &= \{< u,(T_{f_L(e_1,e_2)}(u),I_{f_L(e_1,e_2)}(u),F_{f_L(e_1,e_2)}(u)) >_{u\in G}: f_M(e_2)\in M\} \\ &\quad \text{for } f_L(e_1,e_2)=f_P(e_1)of_M(e_2) \end{aligned}$$

Similarly, for $f_P(e_1)\in P$, the right NSC of M in P is :

$$\begin{aligned} Mof_P(e_1) &= \{f_M(e_2)of_P(e_1): f_M(e_2)\in M\} \\ &= \{< u,(T_{f_Q(e_2,e_1)}(u),I_{f_Q(e_2,e_1)}(u),F_{f_Q(e_2,e_1)}(u)) >_{u\in G}: f_M(e_2)\in M\} \\ &\quad \text{for } f_Q(e_2,e_1)=f_M(e_2)of_P(e_1) \end{aligned}$$

Example 3.1. *Let us consider two NSGs M,P over (G,E) given in Table 1 and Table 2, respectively with respect to t-norm $u*v=uv$ and s-norm $u\diamond v=u+v-uv$ where $G=(\{1,\omega,\omega^2\},\cdot)$ is the multiplicative group of cube root of unity and $E=\{a,b,c\}$.*

Table 1. Table for NSG M.

	$f_M(a)$	$f_M(b)$	$f_M(c)$
1	$(0.5,0.6,0.4)$	$(0.3,0.6,0.7)$	$(0.2,0.3,0.4)$
ω	$(0.4,0.7,0.4)$	$(0.3,0.8,0.6)$	$(0.5,0.2,0.6)$
ω^2	$(0.3,0.8,0.5)$	$(0.4,0.6,0.7)$	$(0.3,0.3,0.4)$

Table 2. Table for NSG P.

	$f_P(a)$	$f_P(b)$	$f_P(c)$
1	$(0.9,0.6,0.2)$	$(0.5,0.4,0.1)$	$(0.4,0.3,0.3)$
ω	$(0.7,0.5,0.4)$	$(0.4,0.5,0.3)$	$(0.5,0.2,0.4)$
ω^2	$(0.8,0.7,0.3)$	$(0.5,0.6,0.2)$	$(0.6,0.2,0.3)$

Clearly, M is a neutrosophic soft subgroup of P. The left NSC of M in P is $\{f_P(a)oM,f_P(b)oM,f_P(c)oM\}$ and it is given by Table 3.

For convenience of Table 3, the result of ω^2 in $f_P(a)of_M(b)$ is provided. $u*v=\min\{u,v\}$ and

$u \diamond v = \max\{u, v\}$ *are considered to prepare Table 3 and Table 4.*

$$\begin{aligned} T_{f_L(a,b)}(\omega^2) &= \max\{T_{f_P(a)}(1) * T_{f_M(b)}(\omega^2), T_{f_P(a)}(\omega^2) * T_{f_M(b)}(1), T_{f_P(a)}(\omega) * T_{f_M(b)}(\omega)\} \\ &= \max\{\min(0.9, 0.4), \min(0.8, 0.3), \min(0.7, 0.3)\} = \max\{0.4, 0.3, 0.3\} = 0.4 \\ I_{f_L(a,b)}(\omega^2) &= \min\{I_{f_P(a)}(1) \diamond I_{f_M(b)}(\omega^2), I_{f_P(a)}(\omega^2) \diamond I_{f_M(b)}(1), I_{f_P(a)}(\omega) \diamond I_{f_M(b)}(\omega)\} \\ &= \min\{\max(0.6, 0.6), \max(0.7, 0.6), \max(0.5, 0.8)\} = \min\{0.6, 0.7, 0.8\} = 0.6 \\ F_{f_L(a,b)}(\omega^2) &= \min\{F_{f_P(a)}(1) \diamond F_{f_M(b)}(\omega^2), F_{f_P(a)}(\omega^2) \diamond F_{f_M(b)}(1), F_{f_P(a)}(\omega) \diamond F_{f_M(b)}(\omega)\} \\ &= \min\{\max(0.2, 0.7), \max(0.3, 0.7), \max(0.4, 0.6)\} = \min\{0.7, 0.7, 0.6\} = 0.6 \end{aligned}$$

Similarly, the right NSC of M in P is $\{Mof_P(a), Mof_P(b), Mof_P(c)\}$ and is given by Table 4.

Table 3. Table for left NSC of M in P

	$f_P(a)of_M(a)$	$f_P(a)of_M(b)$	$f_P(a)of_M(c)$
1	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.5, 0.4)$
ω	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.5, 0.4)$
ω^2	$(0.5, 0.7, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.5, 0.4)$

	$f_P(b)of_M(a)$	$f_P(b)of_M(b)$	$f_P(b)of_M(c)$
1	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.4, 0.4)$
ω	$(0.4, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.4, 0.4)$
ω^2	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.4, 0.4, 0.4)$

	$f_P(c)of_M(a)$	$f_P(c)of_M(b)$	$f_P(c)of_M(c)$
1	$(0.4, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.2, 0.4)$
ω	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.4, 0.3, 0.4)$
ω^2	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.2, 0.4)$

Table 4. Table for right NSC of M in P

	$f_M(a)of_P(a)$	$f_M(b)of_P(a)$	$f_M(c)of_P(a)$
1	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.5, 0.4)$
ω	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.5, 0.4)$
ω^2	$(0.5, 0.7, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.5, 0.4)$

	$f_M(a)of_P(b)$	$f_M(b)of_P(b)$	$f_M(c)of_P(b)$
1	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.4, 0.4)$
ω	$(0.4, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.4, 0.4)$
ω^2	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.4, 0.4, 0.4)$

	$f_M(a)of_P(c)$	$f_M(b)of_P(c)$	$f_M(c)of_P(c)$
1	$(0.4, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.2, 0.4)$
ω	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.4, 0.3, 0.4)$
ω^2	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.6)$	$(0.5, 0.2, 0.4)$

Remark 3.1. *From the above example we see that the nature of coset in an NSG is dissimilar to that in a classical group. We take the case of left NSC only. Similar conclusion holds in case of right NSC also.*

(i) *Any pair of NSCs $f_P(a)oM, f_P(b)oM$ and $f_P(c)oM$ are neither equal nor disjoint. They have only one common element* $f_P(a)of_M(b) = f_P(b)of_M(b) = f_P(c)of_M(b) = \{< 1, (0.4, 0.6, 0.6) >, < \omega, (0.4, 0.6, 0.6) >, < \omega^2, (0.4, 0.6, 0.6) >\}$.

(ii) *For the left NSC of M in P, each element $f_P(a)of_M(a)$, $f_P(b)of_M(a)$, $f_P(c)of_M(a)$, $\cdots \in P \wedge P$ by Definition 2.10, as $f_P(a)of_M(a) \subseteq f_{P\wedge P}(a,a)$, $f_P(b)of_M(a) \subseteq f_{P\wedge P}(b,a)$ and so on. But the cosets do not make partition of $P \wedge P$. Here we use the term 'partition' in the sense that $\cup_e(f_P(e)oM) = P \wedge P$ (i.e., in a particular case it may happen in the Example 3.1 that $f_P(a)oM = f_{P\wedge P}(a,a), f_P(b)oM = f_{P\wedge P}(b,b), f_P(c)oM = f_{P\wedge P}(c,c), \forall f_M(e) \in M$ and so on) and $\forall a,b(a \neq b) \in E, (f_P(a)oM) \cap (f_P(b)oM)$ is not identical with an element of $P \wedge P$. We do not use the expression $(f_P(a)oM) \cap (f_P(b)oM) \notin P \wedge P$ due to the Definition 2.10.*

That is why there is a problem to develop the Lagrange theorem for NSG. So we have introduced here only the concept of Lagrange NSG.

Definition 3.2. Let P be an NSG over (G,E). The number of distinct elements in P is called the order of P and is denoted by $|P|$. A finite NSG P over (G,E) contains finite number of elements, otherwise it is called infinite NSG.

Example 3.2. 1. *Let us consider an NSG P over (V,E) as given in Table 5 where $V = \{e,a,b,c\}$ be the Klein's 4 group and $E = \{\alpha,\beta,\gamma,\delta\}$ be the set of parameters. $*$ and $\diamond$ are $u*v = \max\{u+v-1,0\}, u\diamond v = \min\{u+v,1\}$.*

Table 5. Table for NSG P

	$f_P(\alpha)$	$f_P(\beta)$	$f_P(\gamma)$	$f_P(\delta)$
e	$(0.6,0.3,0.1)$	$(0.8,0.1,0.3)$	$(0.7,0.2,0.1)$	$(0.6,0.3,0.3)$
a	$(0.7,0.2,0.7)$	$(0.5,0.3,0.4)$	$(0.8,0.1,0.2)$	$(0.6,0.3,0.4)$
b	$(0.7,0.2,0.5)$	$(0.8,0.1,0.5)$	$(0.6,0.3,0.3)$	$(0.5,0.4,0.6)$
c	$(0.6,0.3,0.2)$	$(0.7,0.2,0.3)$	$(0.7,0.1,0.4)$	$(0.4,0.2,0.5)$

It is a finite NSG.

2. *Consider another NSG M over (G,E) where $E = \boldsymbol{N}$ (the set of natural numbers), be the parametric set and $G = (\boldsymbol{Z},+)$ be the group of all integers. Define a mapping $f_M : \boldsymbol{N} \to NS(\boldsymbol{Z})$ where, for any $n \in \boldsymbol{N}$ and $x \in \boldsymbol{Z}$,*

$$T_{f_M(n)}(x) = \begin{cases} 0, & x \text{ odd} \\ \frac{1}{5n}, & x \text{ even.} \end{cases} \quad I_{f_M(n)}(x) = \begin{cases} \frac{1}{n+1}, & x \text{ odd} \\ 0, & x \text{ even.} \end{cases} \quad F_{f_M(n)}(x) = \begin{cases} 1-\frac{1}{3n}, & x \text{ odd} \\ 0, & x \text{ even.} \end{cases}$$

Corresponding t-norm and s-norm are taken as $u*v = \min\{u,v\}, u\diamond v = \max\{u,v\}$. It is an infinite NSG.

Definition 3.3. Let P,M be two finite NSGs over (G,E) such that $P \subset M$. If $|P|/|M|$, then P is called a Lagrange neutrosophic soft subgroup. For a finite NSG M, if all it's neutrosophic soft subgroups are Lagrange then M is said to be a Lagrange NSG.

An NSG M, having no Lagrange neutrosophic soft subgroup, is called Lagrange free NSG.

Example 3.3. *We consider the NSG M over $[(\boldsymbol{Z}_3,+),E]$ given in Table 6, where $E = \{e_1,e_2,e_3,e_4\}$. $u*v = \max\{u+v-1,0\}, u\diamond v = \min\{u+v,1\}$ are corresponding t-norm and s-norm.*

We construct two neutrosophic soft subgroups P,Q of M over that $[(\boldsymbol{Z}_3,+),E]$ given in Table 7 and Table 8, respectively.

Here $|P| = 4$ and $|Q| = 3$, as $f_Q(e_2) = f_Q(e_3)$. Thus $|P|/|M|$ but $|Q|$ does not divide $|M|$ i.e., the order of each neutrosophic soft subgroup of M does not divide $|M|$. So in general, an NSG does not satisfy the Lagrange theorem in classical sense. Moreover P is called a Lagrange neutrosophic soft subgroup of M.

Table 6. Table for NSG M

	$f_M(e_1)$	$f_M(e_2)$	$f_M(e_3)$	$f_M(e_4)$
$\overline{0}$	$(0.7, 0.4, 0.3)$	$(0.6, 0.5, 0.6)$	$(0.3, 0.7, 0.4)$	$(0.4, 0.5, 0.5)$
$\overline{1}$	$(0.5, 0.7, 0.2)$	$(0.7, 0.3, 0.4)$	$(0.6, 0.5, 0.3)$	$(0.1, 0.7, 0.6)$
$\overline{2}$	$(0.4, 0.8, 0.5)$	$(0.5, 0.6, 0.7)$	$(0.5, 0.4, 0.2)$	$(0.5, 0.8, 0.4)$

Table 7. Tabular form of neutrosophic soft subgroup P

	$f_P(e_1)$	$f_P(e_2)$	$f_P(e_3)$	$f_P(e_4)$
$\overline{0}$	$(0.6, 0.5, 0.4)$	$(0.5, 0.6, 0.7)$	$(0.2, 0.8, 0.5)$	$(0.3, 0.6, 0.6)$
$\overline{1}$	$(0.4, 0.8, 0.3)$	$(0.6, 0.4, 0.5)$	$(0.5, 0.6, 0.4)$	$(0.1, 0.8, 0.7)$
$\overline{2}$	$(0.3, 0.9, 0.6)$	$(0.4, 0.7, 0.8)$	$(0.4, 0.5, 0.3)$	$(0.4, 0.9, 0.5)$

Table 8. Tabular form of neutrosophic soft subgroup Q

	$f_Q(e_1)$	$f_Q(e_2)$	$f_Q(e_3)$	$f_Q(e_4)$
$\overline{0}$	$(0.5, 0.6, 0.5)$	$(0.3, 0.7, 0.8)$	$(0.3, 0.7, 0.8)$	$(0.4, 0.7, 0.6)$
$\overline{1}$	$(0.3, 0.7, 0.4)$	$(0.5, 0.5, 0.6)$	$(0.5, 0.5, 0.6)$	$(0.1, 0.7, 0.8)$
$\overline{2}$	$(0.1, 0.8, 0.7)$	$(0.2, 0.7, 0.7)$	$(0.2, 0.7, 0.7)$	$(0.3, 0.8, 0.4)$

Theorem 3.1. *Let M, P be two NSGs over (G, E) such that $M \subset P$. Then any two left (right) NSCs of M in P have same cardinality.*

Proof. Proof. Let $f_P(a)oM$ and $f_P(b)oM$ be two left NSCs of M in P over (G, E). We define a neutrosophic soft mapping $(\psi, \xi) : f_P(a)oM \longrightarrow f_P(b)oM$ by $(\psi, \xi)(f_P(a)of_M(e)) = f_P(b)of_M(e'), \forall f_M(e), f_M(e') \in M$ i.e., $(\psi, \xi)(f_L(a, e)) = f_L(b, e')$ where $f_P(a)of_M(e) = f_L(a, e)$. We are to show that (ψ, ξ) is bijective. Let,

$$
\begin{aligned}
& (\psi, \xi)(f_L(a, e_1)) = (\psi, \xi)(f_L(a, e_2)) \\
\Rightarrow\; & T_{f_{\psi(L)}\xi(a,e_1)}(y) = T_{f_{\psi(L)}\xi(a,e_2)}(y),\ I_{f_{\psi(L)}\xi(a,e_1)}(y) = I_{f_{\psi(L)}\xi(a,e_2)}(y), \\
& F_{f_{\psi(L)}\xi(a,e_1)}(y) = F_{f_{\psi(L)}\xi(a,e_2)}(y),\ \forall y \in G. \\
\Rightarrow\; & \max_{\psi(x)} \max_{\xi(a,e_1)} [T_{f_L(a,e_1)}(x)] = \max_{\psi(x)} \max_{\xi(a,e_2)} [T_{f_L(a,e_2)}(x)], \\
& \min_{\psi(x)} \min_{\xi(a,e_1)} [I_{f_L(a,e_1)}(x)] = \min_{\psi(x)} \min_{\xi(a,e_2)} [I_{f_L(a,e_2)}(x)], \\
& \min_{\psi(x)} \min_{\xi(a,e_1)} [F_{f_L(a,e_1)}(x)] = \min_{\psi(x)} \min_{\xi(a,e_2)} [F_{f_L(a,e_2)}(x)],\ \text{ if } x \in \psi^{-1}(y) \\
& \text{ and if } x \notin \psi^{-1}(y),\ \text{the equality is also obvious from definition.} \\
\Rightarrow\; & T_{f_L(a,e_1)}(x) = T_{f_L(a,e_2)}(x),\ I_{f_L(a,e_1)}(x) = I_{f_L(a,e_2)}(x),\ F_{f_L(a,e_1)}(x) = F_{f_L(a,e_2)}(x), \\
& \forall x \in G \text{ (as } x \text{ is arbitrary).} \\
\Rightarrow\; & f_L(a, e_1) = f_L(a, e_2)
\end{aligned}
$$

Thus (ψ, ξ) is injective and from formation (ψ, ξ) is onto also. This ends the proof. □

The theorem can be verified from the Example 3.1.

Definition 3.4. An NSG P over the group G is called abelian NSG if $f_P(a)of_P(b) = f_P(b)of_P(a)$, $\forall a, b \in E$, otherwise it is non-abelian.

Example 3.4. **1.** *The NSG P defined in Example 3.1 is abelian.*

2. *We define another NSG Q over (S_3, E) where S_3 is the group (Table 9) of all permutations on the set $S = \{1, 2, 3\}$ i.e., $S_3 = \{\rho_0(i), \rho_1(123), \rho_2(132), \rho_3(23), \rho_4(13), \rho_5(12)\}$ and $E = \{a, b, c\}$, as given in Table 10.*

Table 9. Table for multiplication operation on S_3

$\cdot$	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
ρ_1	ρ_1	ρ_2	ρ_0	ρ_5	ρ_3	ρ_4
ρ_2	ρ_2	ρ_0	ρ_1	ρ_4	ρ_5	ρ_3
ρ_3	ρ_3	ρ_4	ρ_5	ρ_0	ρ_1	ρ_2
ρ_4	ρ_4	ρ_5	ρ_3	ρ_2	ρ_0	ρ_1
ρ_5	ρ_5	ρ_3	ρ_4	ρ_1	ρ_2	ρ_0

Table 10. Tabular form of NSG Q

	$f_Q(a)$	$f_Q(b)$	$f_Q(c)$
ρ_0	$(0.4, 0.4, 0.9)$	$(0.6, 0.6, 0.6)$	$(0.5, 0.6, 0.6)$
ρ_1	$(0.6, 0.4, 0.7)$	$(0.5, 0.8, 0.5)$	$(0.4, 0.5, 0.7)$
ρ_2	$(0.4, 0.5, 0.8)$	$(0.5, 0.6, 0.7)$	$(0.3, 0.6, 0.4)$
ρ_3	$(0.5, 0.7, 0.6)$	$(0.4, 0.5, 0.4)$	$(0.7, 0.4, 0.5)$
ρ_4	$(0.7, 0.6, 0.7)$	$(0.2, 0.4, 0.8)$	$(0.5, 0.3, 0.4)$
ρ_5	$(0.3, 0.3, 0.5)$	$(0.7, 0.5, 0.3)$	$(0.6, 0.5, 0.6)$

$u * v = \max\{u + v - 1, 0\}, u \diamond v = \min\{u + v, 1\}$ are t-norm and s-norm. Then Q is non-abelian NSG over (S_3, E).

To verify, we estimate the truth membership functions of ρ_1 in $f_Q(a)of_Q(b)$ and $f_Q(b)of_Q(a)$ with respect to the said t-norm. Here $\rho_1 = \rho_0 \cdot \rho_1 = \rho_1 \cdot \rho_0 = \rho_2 \cdot \rho_2 = \rho_3 \cdot \rho_4 = \rho_4 \cdot \rho_5 = \rho_5 \cdot \rho_3$. Now,

$$\begin{aligned}
T_{f_L(a,b)}(\rho_1) &= \max\{T_{f_Q(a)}(\rho_0) * T_{f_Q(b)}(\rho_1), T_{f_Q(a)}(\rho_1) * T_{f_Q(b)}(\rho_0), T_{f_Q(a)}(\rho_2) * T_{f_Q(b)}(\rho_2), \\
&\quad T_{f_Q(a)}(\rho_3) * T_{f_Q(b)}(\rho_4), T_{f_Q(a)}(\rho_4) * T_{f_Q(b)}(\rho_5), T_{f_Q(a)}(\rho_5) * T_{f_Q(b)}(\rho_3)\} \\
&= \max\{0.4 * 0.5, 0.6 * 0.6, 0.4 * 0.5, 0.5 * 0.2, 0.7 * 0.7, 0.3 * 0.4\} \\
&= \max\{0, 0.2, 0, 0, 0.4, 0\} = 0.4 \\
T_{f_L(b,a)}(\rho_1) &= \max\{T_{f_Q(b)}(\rho_0) * T_{f_Q(a)}(\rho_1), T_{f_Q(b)}(\rho_1) * T_{f_Q(a)}(\rho_0), T_{f_Q(b)}(\rho_2) * T_{f_Q(a)}(\rho_2), \\
&\quad T_{f_Q(b)}(\rho_3) * T_{f_Q(a)}(\rho_4), T_{f_Q(b)}(\rho_4) * T_{f_Q(a)}(\rho_5), T_{f_Q(b)}(\rho_5) * T_{f_Q(a)}(\rho_3)\} \\
&= \max\{0.6 * 0.6, 0.5 * 0.4, 0.5 * 0.4, 0.4 * 0.7, 0.2 * 0.3, 0.7 * 0.5\} \\
&= \max\{0.2, 0, 0, 0.1, 0, 0.2\} = 0.2
\end{aligned}$$

Remark 3.2. *An NSG P will be abelian or non-abelian according as the classical group G and the parametric set E over which P is defined are together abelian or non-abelian, respectively unless all neutrosophic soft elements in P are identical.*

Definition 3.5. A neutrosophic soft subgroup M of an NSG P over (G, E) is called neutrosophic normal soft subgroup if $f_P(e)oM = Mof_P(e), \forall f_P(e) \in P$.

Example 3.5. **1.** *From Table 3 and Table 4, it is clear that M is a neutrosophic normal soft subgroup of P over (G,E).*

2. *The null NSG ϕ_G over (G,E) is a neutrosophic normal soft subgroup of every NSG P defined over same (G,E).*

3. *The NSG Q in Table 11 has no neutrosophic normal soft subgroup except ϕ_{S_3}.*

Remark 3.3. **(i)** *For a neutrosophic normal soft subgroup M of an NSG P, each left coset and right coset of M in P are equal. We then call only neutrosophic soft coset of M in P instead of left and right coset separately.*

(ii) *Every neutrosophic soft subgroup of an abelian NSG is always normal as well as abelian. In particular, each abelian NSG is itself normal.*

(iii) *Every non-null neutrosophic soft subgroup of a non-abelian NSG is non-normal.*

(iv) *Every neutrosophic normal soft subgroup of an NSG is abelian. In particular, each neutrosophic normal soft group (N_{NSG}) is itself abelian.*

Theorem 3.2. *Let P be an N_{NSG} over (X,E) and $(\psi,\xi):(X,E)\longrightarrow(Y,E)$ be a neutrosophic soft epimorphism where X,Y are two classical groups and E is a parametric set. Then $(\psi,\xi)(P)$ is an N_{NSG} over (Y,E).*

Proof. Here P is abelian NSG over (X,E) and so both X,E are abelian together. For $x,x_1,x_2\in X$ with $x=x_1\circ x_2$ and $a,b\in E$, we have,

$$
\begin{aligned}
& f_P(a)oP = Pof_P(a),\ \forall f_P(a)\in P\\
\Rightarrow\ & f_P(a)of_P(b) = f_P(b)of_P(a),\ \forall f_P(a),f_P(b)\in P\\
\Rightarrow\ & \max_{x=x_1\circ x_2}[T_{f_P(a)}(x_1)*T_{f_P(b)}(x_2)] = \max_{x=x_1\circ x_2}[T_{f_P(b)}(x_1)*T_{f_P(a)}(x_2)],\\
& \min_{x=x_1\circ x_2}[I_{f_P(a)}(x_1)\diamond I_{f_P(b)}(x_2)] = \min_{x=x_1\circ x_2}[I_{f_P(b)}(x_1)\diamond I_{f_P(a)}(x_2)],\\
& \min_{x=x_1\circ x_2}[F_{f_P(a)}(x_1)\diamond F_{f_P(b)}(x_2)] = \min_{x=x_1\circ x_2}[F_{f_P(b)}(x_1)\diamond F_{f_P(a)}(x_2)].
\end{aligned}
$$

As (ψ,ξ) is a neutrosophic soft epimorphism, for $y,y_1,y_2\in Y$ and $a',b'\in E$ such that $\psi(x)=y,\psi(x_1)=y_1,\psi(x_2)=y_2$ and $\xi(a)=a',\xi(b)=b'$.

Also $\psi(x)=\psi(x_1\circ x_2)=\psi(x_1)\circ\psi(x_2)$ i.e., $y=y_1\circ y_2$. Then,

$$
\begin{aligned}
& \max_{y=y_1\circ y_2}[T_{f_{\psi(P)}(a')}(y_1)*T_{f_{\psi(P)}(b')}(y_2)]\\
=\ & \max_{y=y_1\circ y_2}\{\max_{y_1=\psi(x_1)}\max_{a'=\xi(a)}[T_{f_P(a)}(x_1)]*\max_{y_2=\psi(x_2)}\max_{b'=\xi(b)}[T_{f_P(b)}(x_2)]\}\\
=\ & \max_{y=y_1\circ y_2}\{\max_{\psi(x_1)\circ\psi(x_2)}\max_{\xi(a)\circ\xi(b)}[T_{f_P(a)}(x_1)*T_{f_P(b)}(x_2)]\}\\
=\ & \max_{\psi(x_1)\circ\psi(x_2)}\max_{\xi(a)\circ\xi(b)}\{\max_{x=x_1\circ x_2}[T_{f_P(a)}(x_1)*T_{f_P(b)}(x_2)]\}\\
=\ & \max_{\psi(x_1\circ x_2)}\max_{\xi(a\circ b)}\{\max_{x=x_1\circ x_2}[T_{f_P(b)}(x_1)*T_{f_P(a)}(x_2)]\}
\end{aligned}
$$

by the condition of truth membership function provided

$$
\begin{aligned}
&= \max_{\psi(x_1 \circ x_2)} \max_{\xi(b \circ a)} \{ \max_{x=x_1 \circ x_2} [T_{f_P(b)}(x_1) * T_{f_P(a)}(x_2)] \} \\
&\quad (\text{ as } E \text{ is abelian in classical sense}) \\
&= \max_{y=y_1 \circ y_2} \{ \max_{\psi(x_1 \circ x_2)} \max_{\xi(b \circ a)} [T_{f_P(b)}(x_1) * T_{f_P(a)}(x_2)] \} \\
&= \max_{y=y_1 \circ y_2} \{ \max_{\psi(x_1) \circ \psi(x_2)} \max_{\xi(b) \circ \xi(a)} [T_{f_P(b)}(x_1) * T_{f_P(a)}(x_2)] \} \\
&= \max_{y=y_1 \circ y_2} \{ \max_{y_1=\psi(x_1)} \max_{b'=\xi(b)} [T_{f_P(b)}(x_1)] * \max_{y_2=\psi(x_2)} \max_{a'=\xi(a)} [T_{f_P(a)}(x_2)] \} \\
&= \max_{y=y_1 \circ y_2} [T_{f_{\psi(P)}(b')}(y_1) * T_{f_{\psi(P)}(a')}(y_2)]. \qquad (1)
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\min_{y=y_1 \circ y_2} [I_{f_{\psi(P)}(a')}(y_1) \diamond I_{f_{\psi(P)}(b')}(y_2)] \\
&= \min_{y=y_1 \circ y_2} \{ \min_{y_1=\psi(x_1)} \min_{a'=\xi(a)} [I_{f_P(a)}(x_1)] \diamond \min_{y_2=\psi(x_2)} \min_{b'=\xi(b)} [I_{f_P(b)}(x_2)] \} \\
&= \min_{y=y_1 \circ y_2} \{ \min_{\psi(x_1) \circ \psi(x_2)} \min_{\xi(a) \circ \xi(b)} [I_{f_P(a)}(x_1) \diamond I_{f_P(b)}(x_2)] \} \\
&= \min_{\psi(x_1) \circ \psi(x_2)} \min_{\xi(a) \circ \xi(b)} \{ \min_{x=x_1 \circ x_2} [I_{f_P(a)}(x_1) \diamond I_{f_P(b)}(x_2)] \} \\
&= \min_{\psi(x_1 \circ x_2)} \min_{\xi(b \circ a)} \{ \min_{x=x_1 \circ x_2} [I_{f_P(b)}(x_1) \diamond I_{f_P(a)}(x_2)] \} \\
&= \min_{\psi(x_1 \circ x_2)} \min_{\xi(b \circ a)} \{ \min_{x=x_1 \circ x_2} [I_{f_P(b)}(x_1) \diamond I_{f_P(a)}(x_2)] \} \\
&= \min_{y=y_1 \circ y_2} \{ \min_{\psi(x_1 \circ x_2)} \min_{\xi(b \circ a)} [I_{f_P(b)}(x_1) \diamond I_{f_P(a)}(x_2)] \} \\
&= \min_{y=y_1 \circ y_2} \{ \min_{\psi(x_1) \circ \psi(x_2)} \min_{\xi(b) \circ \xi(a)} [I_{f_P(b)}(x_1) \diamond I_{f_P(a)}(x_2)] \} \\
&= \min_{y=y_1 \circ y_2} \{ \min_{y_1=\psi(x_1)} \min_{b'=\xi(b)} [I_{f_P(b)}(x_1)] \diamond \min_{y_2=\psi(x_2)} \min_{a'=\xi(a)} [I_{f_P(a)}(x_2)] \} \\
&= \min_{y=y_1 \circ y_2} [I_{f_{\psi(P)}(b')}(y_1) \diamond I_{f_{\psi(P)}(a')}(y_2)] \qquad (2)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\min_{y=y_1 \circ y_2} [F_{f_{\psi(P)}(a')}(y_1) \diamond F_{f_{\psi(P)}(b')}(y_2)] \\
&= \min_{y=y_1 \circ y_2} \{ \min_{y_1=\psi(x_1)} \min_{a'=\xi(a)} [F_{f_P(a)}(x_1)] \diamond \min_{y_2=\psi(x_2)} \min_{b'=\xi(b)} [F_{f_P(b)}(x_2)] \} \\
&= \min_{y=y_1 \circ y_2} \{ \min_{\psi(x_1) \circ \psi(x_2)} \min_{\xi(a) \circ \xi(b)} [F_{f_P(a)}(x_1) \diamond F_{f_P(b)}(x_2)] \} \\
&= \min_{\psi(x_1) \circ \psi(x_2)} \min_{\xi(a) \circ \xi(b)} \{ \min_{x=x_1 \circ x_2} [F_{f_P(a)}(x_1) \diamond F_{f_P(b)}(x_2)] \} \\
&= \min_{\psi(x_1 \circ x_2)} \min_{\xi(b \circ a)} \{ \min_{x=x_1 \circ x_2} [F_{f_P(b)}(x_1) \diamond F_{f_P(a)}(x_2)] \} \\
&= \min_{\psi(x_1 \circ x_2)} \min_{\xi(b \circ a)} \{ \min_{x=x_1 \circ x_2} [F_{f_P(b)}(x_1) \diamond F_{f_P(a)}(x_2)] \}
\end{aligned}
$$

$$\begin{aligned}
&= \min_{y=y_1\circ y_2}\{\min_{\psi(x_1\circ x_2)}\min_{\xi(b\circ a)}[F_{f_P(b)}(x_1)\diamond F_{f_P(a)}(x_2)]\}\\
&= \min_{y=y_1\circ y_2}\{\min_{\psi(x_1)\circ\psi(x_2)}\min_{\xi(b)\circ\xi(a)}[F_{f_P(b)}(x_1)\diamond F_{f_P(a)}(x_2)]\}\\
&= \min_{y=y_1\circ y_2}\{\min_{y_1=\psi(x_1)}\min_{b'=\xi(b)}[F_{f_P(b)}(x_1)]\diamond \min_{y_2=\psi(x_2)}\min_{a'=\xi(a)}[F_{f_P(a)}(x_2)]\}\\
&= \min_{y=y_1\circ y_2}[F_{f_{\psi(P)}(b')}(y_1)\diamond F_{f_{\psi(P)}(a')}(y_2)]. \qquad (3)
\end{aligned}$$

From (1), (2) and (3), we have, $f_{\psi(P)}(a')of_{\psi(P)}(b') = f_{\psi(P)}(b')of_{\psi(P)}(a')$

$\Rightarrow f_{\psi(P)}(a')o(\psi,\xi)(P) = (\psi,\xi)(P)of_{\psi(P)}(a')$, as $f_{\psi(P)}(b') \in (\psi,\xi)(P)$ arbitrary.

As $f_{\psi(P)}(a') \in (\psi,\xi)(P)$ is an arbitrary, so $(\psi,\xi)(P)$ is an N_{NSG} over (Y,E). □

Theorem 3.3. *Let M be an N_{NSG} over (Y,E) and $(\psi,\xi):(X,E)\longrightarrow(Y,E)$ be a neutrosophic soft homomorphism where X,Y are two classical groups and E is a parametric set. Then $(\psi,\xi)^{-1}(M)$ is an N_{NSG} over (X,E).*

Proof. Here M is abelian NSG over (Y,E) and so both Y,E are abelian together. Let $a,b\in\xi^{-1}(E)$ and $x,x_1,x_2\in X$ with $x=x_1\circ x_2$. As (ψ,ξ) is a neutrosophic soft homomorphism, so $\psi(x)=\psi(x_1\circ x_2)=\psi(x_1)\circ\psi(x_2)$. Then,

$$\begin{aligned}
& f_M(\xi(a))oM = Mof_M(\xi(a)),\ \forall f_M(\xi(a))\in M\\
\Rightarrow\ & f_M(\xi(a))of_M(\xi(b)) = f_M(\xi(b))of_M(\xi(a)),\ \forall f_M(\xi(a)), f_M(\xi(b))\in M\\
\Rightarrow\ & \max_{\psi(x)=\psi(x_1)\circ\psi(x_2)}[T_{f_M[\xi(a)]}(\psi(x_1)) * T_{f_M[\xi(b)]}(\psi(x_2))]\\
& = \max_{\psi(x)=\psi(x_1)\circ\psi(x_2)}[T_{f_M[\xi(b)]}(\psi(x_1)) * T_{f_M[\xi(a)]}(\psi(x_2))],\\
& \min_{\psi(x)=\psi(x_1)\circ\psi(x_2)}[I_{f_M[\xi(a)]}(\psi(x_1)) \diamond I_{f_M[\xi(b)]}(\psi(x_2))]\\
& = \min_{\psi(x)=\psi(x_1)\circ\psi(x_2)}[I_{f_M[\xi(b)]}(\psi(x_1)) \diamond I_{f_M[\xi(a)]}(\psi(x_2))],\\
& \min_{\psi(x)=\psi(x_1)\circ\psi(x_2)}[F_{f_M[\xi(a)]}(\psi(x_1)) \diamond F_{f_M[\xi(b)]}(\psi(x_2))]\\
& = \min_{\psi(x)=\psi(x_1)\circ\psi(x_2)}[F_{f_M[\xi(b)]}(\psi(x_1)) \diamond F_{f_M[\xi(a)]}(\psi(x_2))].
\end{aligned}$$

As (ψ,ξ) is a neutrosophic soft homomorphism, so $\psi(x)=\psi(x_1\circ x_2)=\psi(x_1)\circ\psi(x_2)$.

$$\begin{aligned}
\text{Now,}\quad & \max_{x=x_1\circ x_2}[T_{f_{\psi^{-1}(M)}(a)}(x_1) * T_{f_{\psi^{-1}(M)}(b)}(x_2)]\\
&= \max_{\psi(x)=\psi(x_1)\circ\psi(x_2)}[T_{f_M[\xi(a)]}(\psi(x_1)) * T_{f_M[\xi(b)]}(\psi(x_2))]\\
&= \max_{\psi(x)=\psi(x_1\circ x_2)}[T_{f_M[\xi(b)]}(\psi(x_1)) * T_{f_M[\xi(a)]}(\psi(x_2))]\\
&\qquad \text{(by the condition of truth membership function provided)}\\
&= \max_{x=x_1\circ x_2}[T_{f_{\psi^{-1}(M)}(b)}(x_1) * T_{f_{\psi^{-1}(M)}(a)}(x_2)]. \qquad (4)
\end{aligned}$$

$$\text{Next,} \quad \min_{x=x_1\circ x_2}[I_{f_{\psi^{-1}(M)}(a)}(x_1)\diamond I_{f_{\psi^{-1}(M)}(b)}(x_2)]$$
$$= \min_{\psi(x)=\psi(x_1)\circ\psi(x_2)}[I_{f_M[\xi(a)]}(\psi(x_1))\diamond I_{f_M[\xi(b)]}(\psi(x_2))]$$
$$= \min_{\psi(x)=\psi(x_1\circ x_2)}[I_{f_M[\xi(b)]}(\psi(x_1))\diamond I_{f_M[\xi(a)]}(\psi(x_2))]$$
(by the condition of indeterminacy membership function provided)
$$= \min_{x=x_1\circ x_2}[I_{f_{\psi^{-1}(M)}(b)}(x_1)\diamond I_{f_{\psi^{-1}(M)}(a)}(x_2)]. \tag{5}$$

$$\text{Finally,} \quad \min_{x=x_1\circ x_2}[F_{f_{\psi^{-1}(M)}(a)}(x_1)\diamond F_{f_{\psi^{-1}(M)}(b)}(x_2)]$$
$$= \min_{\psi(x)=\psi(x_1)\circ\psi(x_2)}[F_{f_M[\xi(a)]}(\psi(x_1))\diamond F_{f_M[\xi(b)]}(\psi(x_2))]$$
$$= \min_{\psi(x)=\psi(x_1\circ x_2)}[F_{f_M[\xi(b)]}(\psi(x_1))\diamond F_{f_M[\xi(a)]}(\psi(x_2))]$$
$$= \min_{x=x_1\circ x_2}[F_{f_{\psi^{-1}(M)}(b)}(x_1)\diamond F_{f_{\psi^{-1}(M)}(a)}(x_2)]. \tag{6}$$

From (4), (5) and (6), we have,

$f_{\psi^{-1}(M)}(a) o f_{\psi^{-1}(M)}(b) = f_{\psi^{-1}(M)}(b) o f_{\psi^{-1}(M)}(a)$.

$\Rightarrow f_{\psi^{-1}(M)}(a) o (\psi,\xi)^{-1}(M) = (\psi,\xi)^{-1}(M) o f_{\psi^{-1}(M)}(a)$. [as $f_{\psi^{-1}(M)}(b) \in (\psi,\xi)^{-1}(M)$ arbitrary.]

As $f_{\psi^{-1}(M)}(a) \in (\psi,\xi)^{-1}(M)$ is an arbitrary, so $(\psi,\xi)^{-1}(M)$ is an N_{NSG} over (X,E). □

Theorem 3.4. *Let M be a neutrosophic normal soft subgroup of an NSG P over (G,E) and Ω be a collection of all distinct NSCs of M in P. Then Ω forms an NSG over $(G,E\times E)$.*

Proof. As M is an N_{NSG} defined over (G,E), then M is abelian and so G is abelian by Remark 3.2. Moreover there is no distinction between left and right NSC of M in P over $(G,E\times E)$. Let $(a,b)\in E\times E$ and $x,y,z,y_1,y_2,z_1,z_2\in G$ be arbitrary such that $x=y\circ z, y=y_1\circ y_2, z=z_1\circ z_2$. Suppose $PoM=L$. Now,

$$T_{f_L(a,b)}(y) * T_{f_L(a,b)}(z)$$
$$= \max_{y=y_1\circ y_2}[T_{f_P(a)}(y_1) * T_{f_M(b)}(y_2)] * \max_{z=z_1\circ z_2}[T_{f_P(a)}(z_1) * T_{f_M(b)}(z_2)]$$
$$= \max_{x=(y_1\circ y_2)\circ(z_1\circ z_2)}[(T_{f_P(a)}(y_1) * T_{f_M(b)}(y_2)) * (T_{f_P(a)}(z_1) * T_{f_M(b)}(z_2))]$$
$$= \max_{x=(y_1\circ y_2)\circ(z_1\circ z_2)}[(T_{f_P(a)}(y_1) * T_{f_P(a)}(z_1)) * (T_{f_M(b)}(y_2) * T_{f_M(b)}(z_2))]$$
(as $*$ is commutative)
$$\leq \max_{x=(y_1\circ y_2)\circ(z_1\circ z_2)}[T_{f_P(a)}(y_1\circ z_1) * T_{f_M(b)}(y_2\circ z_2)]$$
(as M and P are two neutrosophic soft groups)
$$= T_{f_L(a,b)}[(y_1\circ z_1)\circ(y_2\circ z_2)]$$
$$= T_{f_L(a,b)}[(y_1\circ y_2)\circ(z_1\circ z_2)] \quad (\text{ as } G \text{ is abelian })$$
$$= T_{f_L(a,b)}(y\circ z).$$

Thus, $T_{f_L(a,b)}(y\circ z) \geq T_{f_L(a,b)}(y) * T_{f_L(a,b)}(z)$. Next,

$$\begin{aligned}
& I_{f_{L(a,b)}}(y) * I_{f_{L(a,b)}}(z) \\
= & \min_{y=y_1\circ y_2}[I_{f_P(a)}(y_1) \diamond I_{f_M(b)}(y_2)] \diamond \min_{z=z_1\circ z_2}[I_{f_P(a)}(z_1) \diamond I_{f_M(b)}(z_2)] \\
= & \min_{x=(y_1\circ y_2)\circ(z_1\circ z_2)}[(I_{f_P(a)}(y_1) \diamond I_{f_M(b)}(y_2)) \diamond (I_{f_P(a)}(z_1) \diamond I_{f_M(b)}(z_2))] \\
= & \min_{x=(y_1\circ y_2)\circ(z_1\circ z_2)}[(I_{f_P(a)}(y_1) \diamond I_{f_P(a)}(z_1)) \diamond (I_{f_M(b)}(y_2) \diamond I_{f_M(b)}(z_2))] \\
\geq & \min_{x=(y_1\circ y_2)\circ(z_1\circ z_2)}[I_{f_P(a)}(y_1 \circ z_1) \diamond I_{f_M(b)}(y_2 \circ z_2)] \\
= & I_{f_{L(a,b)}}[(y_1 \circ z_1) \circ (y_2 \circ z_2)] = I_{f_{L(a,b)}}[(y_1 \circ y_2) \circ (z_1 \circ z_2)] = I_{f_{L(a,b)}}(y \circ z).
\end{aligned}$$

So, $I_{f_{L(a,b)}}(y \circ z) \leq I_{f_{L(a,b)}}(y) \diamond I_{f_{L(a,b)}}(z)$. Finally,

$$\begin{aligned}
& F_{f_{L(a,b)}}(y) \diamond F_{f_{L(a,b)}}(z) \\
= & \min_{y=y_1\circ y_2}[F_{f_P(a)}(y_1) \diamond F_{f_M(b)}(y_2)] \diamond \min_{z=z_1\circ z_2}[F_{f_P(a)}(z_1) \diamond F_{f_M(b)}(z_2)] \\
= & \min_{x=(y_1\circ y_2)\circ(z_1\circ z_2)}[(F_{f_P(a)}(y_1) \diamond F_{f_M(b)}(y_2)) \diamond (F_{f_P(a)}(z_1) \diamond F_{f_M(b)}(z_2))] \\
= & \min_{x=(y_1\circ y_2)\circ(z_1\circ z_2)}[(F_{f_P(a)}(y_1) \diamond F_{f_P(a)}(z_1)) \diamond (F_{f_M(b)}(y_2) \diamond F_{f_M(b)}(z_2))] \\
\geq & \min_{x=(y_1\circ y_2)\circ(z_1\circ z_2)}[F_{f_P(a)}(y_1 \circ z_1) \diamond F_{f_M(b)}(y_2 \circ z_2)] \\
= & F_{f_{L(a,b)}}[(y_1 \circ z_1) \circ (y_2 \circ z_2)] = F_{f_{L(a,b)}}[(y_1 \circ y_2) \circ (z_1 \circ z_2)] = F_{f_{L(a,b)}}(y \circ z)
\end{aligned}$$

Thus $F_{f_{L(a,b)}}(y \circ z) \leq F_{f_{L(a,b)}}(y) \diamond F_{f_{L(a,b)}}(z)$ and this ends the proof. □

This group Ω is called neutrosophic soft quotient group of P by M over $(G, E \times E)$ and is denoted by P/M.

Proposition 3.1. *Let M be a neutrosophic normal soft subgroup of an NSG P over (G,E). Then there exists a neutrosophic soft homomorphism $(\psi,\xi) : P \longrightarrow P/M$ defined as $(\psi,\xi)(f_P(a)) = f_P(a)oM, \forall f_P(a) \in P$ if $u * v = \min\{u,v\}$ and $u \diamond v = \max\{u,v\}$.*

Proof. Let $(\psi,\xi) : P \longrightarrow P/M$ be defined as $(\psi,\xi)(f_P(a)) = f_P(a)of_M(e), \forall f_M(e) \in M$. We shall show (ψ,ξ) a neutrosophic soft homomorphism in the sense that

$$\begin{aligned}
& (\psi,\xi)[f_P(a)of_P(b)] = (\psi,\xi)[f_P(a)]o(\psi,\xi)[f_P(b)], \forall a,b \in E. \\
\Rightarrow\ & (f_P(a)of_P(b))of_M(e) = (f_P(a)of_M(e))o(f_P(b)of_M(e)).
\end{aligned}$$

Since M is an N_{NSG} over (G,E), so G is commutative. Let $x, y, z, s_1, s_2, s, t \in G$ such that $t = x \circ y, s_1 =$

$x \circ z, s_2 = y \circ z, s = t \circ z$. Then,

$$\begin{aligned}
f_P(a)of_P(b) &= \{< t, \max_{t=x\circ y}[T_{f_P(a)}(x) * T_{f_P(b)}(y)], \min_{t=x\circ y}[I_{f_P(a)}(x) \diamond I_{f_P(b)}(y)],\\
&\quad \min_{t=x\circ y}[F_{f_P(a)}(x) \diamond F_{f_P(b)}(y)] >: t \in G\}\\
(f_P(a)of_P(b))of_M(e) &= \{< s, \max_{s=t\circ z}[\max_{t=x\circ y}[T_{f_P(a)}(x) * T_{f_P(b)}(y)] * T_{f_M(e)}(z)],\\
&\quad \min_{s=t\circ z}[\min_{t=x\circ y}[I_{f_P(a)}(x) \diamond I_{f_P(b)}(y)] \diamond I_{f_M(e)}(z)],\\
&\quad \min_{s=t\circ z}[\min_{t=x\circ y}[F_{f_P(a)}(x) \diamond F_{f_P(b)}(y)] \diamond F_{f_M(e)}(z)] >: s \in G\}
\end{aligned}$$

$$\begin{aligned}
f_P(a)of_M(e) &= \{< s_1, \max_{x\circ z}[T_{f_P(a)}(x) * T_{f_M(e)}(z)], \min_{x\circ z}[I_{f_P(a)}(x) \diamond I_{f_M(e)}(z)],\\
&\quad \min_{x\circ z}[F_{f_N(a)}(x) \diamond F_{f_M(e)}(z)] >: s_1 \in G\}\\
f_P(b)of_M(e) &= \{< s_2, \max_{y\circ z}[T_{f_P(b)}(y) * T_{f_M(e)}(z)], \min_{y\circ z}[I_{f_P(b)}(y) \diamond I_{f_M(e)}(z)],\\
&\quad \min_{y\circ z}[F_{f_P(b)}(y) \diamond F_{f_M(e)}(z)] >: s_2 \in G\}
\end{aligned}$$

Now, $\max_{s_1\circ s_2}\{\max_{s_1=x\circ z}[T_{f_P(a)}(x) * T_{f_M(e)}(z)] * \max_{s_2=y\circ z}[T_{f_P(b)}(y) * T_{f_M(e)}(z)]\}$

$$\begin{aligned}
&= \max_{s_1\circ s_2}\{\max_{s_1\circ s_2=(x\circ z)\circ(y\circ z)}([T_{f_P(a)}(x) * T_{f_M(e)}(z)] * [T_{f_P(b)}(y) * T_{f_M(e)}(z)])\}\\
&= \max_{s_1\circ s_2}\{\max_{s_1\circ s_2=(x\circ y)\circ(z\circ z)}([T_{f_P(a)}(x) * T_{f_P(b)}(y)] * [T_{f_M(e)}(z) * T_{f_M(e)}(z)])\}
\end{aligned}$$

(as $*$ and G both are commutative)

$$= \max_{s=t\circ z}\{\max_{t=x\circ y}[T_{f_P(a)}(x) * T_{f_P(b)}(y)] * T_{f_M(e)}(z)\} \tag{7}$$

Next, $\min_{s_1\circ s_2}\{\min_{s_1=x\circ z}[I_{f_P(a)}(x) \diamond I_{f_M(e)}(z)] \diamond \max_{s_2=y\circ z}[I_{f_P(b)}(y) \diamond I_{f_M(e)}(z)]\}$

$$\begin{aligned}
&= \min_{s_1\circ s_2}\{\min_{s_1\circ s_2=(x\circ z)\circ(y\circ z)}([I_{f_P(a)}(x) \diamond I_{f_M(e)}(z)] \diamond [I_{f_P(b)}(y) \diamond I_{f_M(e)}(z)])\}\\
&= \min_{s_1\circ s_2}\{\min_{s_1\circ s_2=(x\circ y)\circ(z\circ z)}([I_{f_P(a)}(x) \diamond I_{f_P(b)}(y)] \diamond [I_{f_M(e)}(z) \diamond I_{f_M(e)}(z)])\}
\end{aligned}$$

$$= \min_{s=t\circ z}\{\min_{t=x\circ y}[I_{f_P(a)}(x) \diamond I_{f_P(b)}(y)] \diamond I_{f_M(e)}(z)\} \tag{8}$$

Finally, $\min_{s_1\circ s_2}\{\min_{s_1=x\circ z}[F_{f_P(a)}(x) \diamond F_{f_M(e)}(z)] \diamond \max_{s_2=y\circ z}[F_{f_P(b)}(y) \diamond F_{f_M(e)}(z)]\}$

$$\begin{aligned}
&= \min_{s_1\circ s_2}\{\min_{s_1\circ s_2=(x\circ z)\circ(y\circ z)}([F_{f_P(a)}(x) \diamond F_{f_M(e)}(z)] \diamond [F_{f_P(b)}(y) \diamond F_{f_M(e)}(z)])\}\\
&= \min_{s_1\circ s_2}\{\min_{s_1\circ s_2=(x\circ y)\circ(z\circ z)}([F_{f_P(a)}(x) \diamond F_{f_P(b)}(y)] \diamond [F_{f_M(e)}(z) \diamond F_{f_M(e)}(z)])\}
\end{aligned}$$

$$= \min_{s=t\circ z}\{\min_{t=x\circ y}[F_{f_P(a)}(x) \diamond F_{f_P(b)}(y)] \diamond F_{f_M(e)}(z)\} \tag{9}$$

From (7), (8) and (9), we see that

$(f_P(a)of_M(e))o(f_P(b)of_M(e)) = (f_P(a)of_P(b))of_M(e)$ and this ends the proof. □

Definition 3.6. Let M and P be two NSGs over (G,E). Their direct product is denoted by $M\otimes P$ and is defined as :

$$\begin{aligned} M\otimes P &= \{\langle (a,b),(f_M(a),f_P(b))\rangle : (a,b)\in E\times E\} \text{ where} \\ (f_M(a),f_N(b)) &= \{< (x,y), T_{f_{M(a)}}(x) * T_{f_{P(b)}}(y), I_{f_{M(a)}}(x) \diamond I_{f_{P(b)}}(y), \\ &\qquad F_{f_{M(a)}}(x) \diamond F_{f_{P(b)}}(y) >: (x,y)\in G\times G\} \end{aligned}$$

The definition can be extended for any finite number of NSGs.

Example 3.6. *Consider NSGs M,P defined in Table 1 and Table 2, respectively. Then their direct product $M\otimes P$ is given in Table 11. The $*$ and $\diamond$ are taken as $u*v=\min\{u,v\}, u\diamond v=\max\{u,v\}$.*

Theorem 3.5. *Let M,P be two NSGs over (G,E). Then their direct product $M\otimes P$ is also an NSG over $(G\times G, E\times E)$.*

Proof. Let $(a,b)\in E\times E$ and $(x,y)\in G\times G$ be arbitrary such that $(x,y)=(x_1,y_1)\circ(x_2,y_2)$ for $x_1,x_2,y_1,y_2\in G$. Now,

$$\begin{aligned} T_{f_{M\otimes P(a,b)}}[(x_1,y_1)\circ(x_2,y_2)] &= T_{f_{M\otimes P(a,b)}}(x_1\circ x_2, y_1\circ y_2) \\ &= T_{f_{M(a)}}(x_1\circ x_2) * T_{f_{P(b)}}(y_1\circ y_2) \\ &\geq [T_{f_{M(a)}}(x_1) * T_{f_{M(a)}}(x_2)] * [T_{f_{P(b)}}(y_1) * T_{f_{P(b)}}(y_2)] \\ &\quad (\text{ as } M,P \text{ are two NSGs }) \\ &= [T_{f_{M(a)}}(x_1) * T_{f_{P(b)}}(y_1)] * [T_{f_{M(a)}}(x_2) * T_{f_{P(b)}}(y_2)] \\ &\qquad (\text{ as } * \text{ is commutative }) \\ &= T_{f_{M\otimes P(a,b)}}(x_1,y_1) * T_{f_{M\otimes N(a,b)}}(x_2,y_2) \qquad (10) \\ I_{f_{M\otimes P(a,b)}}[(x_1,y_1)\circ(x_2,y_2)] &= I_{f_{M\otimes P(a,b)}}(x_1\circ x_2, y_1\circ y_2) \\ &= I_{f_{M(a)}}(x_1\circ x_2) \diamond I_{f_{P(b)}}(y_1\circ y_2) \\ &\leq [I_{f_{M(a)}}(x_1) \diamond I_{f_{M(a)}}(x_2)] \diamond [I_{f_{P(b)}}(y_1) \diamond I_{f_{P(b)}}(y_2)] \\ &= [I_{f_{M(a)}}(x_1) \diamond I_{f_{P(b)}}(y_1)] \diamond [I_{f_{M(a)}}(x_2) \diamond I_{f_{P(b)}}(y_2)] \\ &= I_{f_{M\otimes N(a,b)}}(x_1,y_1) \diamond I_{f_{M\otimes P(a,b)}}(x_2,y_2) \qquad (11) \\ F_{f_{M\otimes P(a,b)}}[(x_1,y_1)\circ(x_2,y_2)] &= F_{f_{M\otimes P(a,b)}}(x_1\circ x_2, y_1\circ y_2) \\ &= F_{f_{M(a)}}(x_1\circ x_2) \diamond F_{f_{P(b)}}(y_1\circ y_2) \\ &\leq [F_{f_{M(a)}}(x_1) \diamond F_{f_{M(a)}}(x_2)] \diamond [F_{f_{P(b)}}(y_1) \diamond F_{f_{P(b)}}(y_2)] \\ &= [F_{f_{M(a)}}(x_1) \diamond F_{f_{P(b)}}(y_1)] \diamond [F_{f_{M(a)}}(x_2) \diamond F_{f_{P(b)}}(y_2)] \\ &= F_{f_{M\otimes P(a,b)}}(x_1,y_1) \diamond F_{f_{M\otimes P(a,b)}}(x_2,y_2) \qquad (12) \end{aligned}$$

Hence the theorem follows from (10), (11) and (12).
It can be easily verified from Table 11 taking $u*v=uv$ and $u\diamond v=u+v-uv$. □

Table 11. Table for direct product $M \otimes P$

	$(f_M(a), f_P(a))$	$(f_M(a), f_P(b))$	$(f_M(a), f_P(c))$
$(1,1)$	$(0.5, 0.6, 0.4)$	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.4)$
$(1,\omega)$	$(0.5, 0.6, 0.4)$	$(0.4, 0.6, 0.4)$	$(0.5, 0.6, 0.4)$
$(1,\omega^2)$	$(0.5, 0.7, 0.4)$	$(0.5, 0.6, 0.4)$	$(0.5, 0.6, 0.4)$
$(\omega,1)$	$(0.4, 0.7, 0.4)$	$(0.4, 0.7, 0.4)$	$(0.4, 0.7, 0.4)$
(ω,ω)	$(0.4, 0.7, 0.4)$	$(0.4, 0.7, 0.4)$	$(0.4, 0.7, 0.4)$
(ω,ω^2)	$(0.4, 0.7, 0.4)$	$(0.4, 0.7, 0.4)$	$(0.4, 0.7, 0.4)$
$(\omega^2,1)$	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 0.5)$
(ω^2,ω)	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 0.5)$
(ω^2,ω^2)	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 0.5)$	$(0.3, 0.8, 0.5)$
	$(f_M(b), f_P(a))$	$(f_M(b), f_P(b))$	$(f_M(b), f_P(c))$
$(1,1)$	$(0.3, 0.6, 0.7)$	$(0.3, 0.6, 0.7)$	$(0.3, 0.6, 0.7)$
$(1,\omega)$	$(0.3, 0.6, 0.7)$	$(0.3, 0.6, 0.7)$	$(0.3, 0.6, 0.7)$
$(1,\omega^2)$	$(0.3, 0.7, 0.7)$	$(0.3, 0.6, 0.7)$	$(0.3, 0.6, 0.7)$
$(\omega,1)$	$(0.3, 0.8, 0.6)$	$(0.3, 0.8, 0.6)$	$(0.3, 0.8, 0.6)$
(ω,ω)	$(0.3, 0.8, 0.6)$	$(0.3, 0.8, 0.6)$	$(0.3, 0.8, 0.6)$
(ω,ω^2)	$(0.3, 0.8, 0.6)$	$(0.3, 0.8, 0.6)$	$(0.3, 0.8, 0.6)$
$(\omega^2,1)$	$(0.4, 0.6, 0.7)$	$(0.4, 0.6, 0.7)$	$(0.4, 0.6, 0.7)$
(ω^2,ω)	$(0.4, 0.6, 0.7)$	$(0.4, 0.6, 0.7)$	$(0.4, 0.6, 0.7)$
(ω^2,ω^2)	$(0.4, 0.7, 0.7)$	$(0.4, 0.6, 0.7)$	$(0.4, 0.6, 0.7)$
	$(f_M(c), f_P(a))$	$(f_M(c), f_P(b))$	$(f_M(c), f_P(c))$
$(1,1)$	$(0.2, 0.6, 0.4)$	$(0.2, 0.4, 0.4)$	$(0.2, 0.3, 0.4)$
$(1,\omega)$	$(0.2, 0.5, 0.4)$	$(0.2, 0.5, 0.4)$	$(0.2, 0.3, 0.4)$
$(1,\omega^2)$	$(0.2, 0.7, 0.4)$	$(0.2, 0.6, 0.4)$	$(0.2, 0.3, 0.4)$
$(\omega,1)$	$(0.5, 0.6, 0.6)$	$(0.5, 0.4, 0.6)$	$(0.4, 0.3, 0.6)$
(ω,ω)	$(0.5, 0.5, 0.6)$	$(0.4, 0.5, 0.6)$	$(0.5, 0.2, 0.6)$
(ω,ω^2)	$(0.5, 0.7, 0.6)$	$(0.5, 0.6, 0.6)$	$(0.5, 0.2, 0.6)$
$(\omega^2,1)$	$(0.3, 0.6, 0.4)$	$(0.3, 0.4, 0.4)$	$(0.3, 0.3, 0.4)$
(ω^2,ω)	$(0.3, 0.5, 0.4)$	$(0.3, 0.5, 0.4)$	$(0.3, 0.3, 0.4)$
(ω^2,ω^2)	$(0.3, 0.7, 0.4)$	$(0.3, 0.6, 0.4)$	$(0.3, 0.3, 0.4)$

Definition 3.7. An NSG over (G, E) is said to be a simple NSG if it has no neutrosophic normal soft subgroup other than ϕ_G.

Example 3.7. 1. *In the Example 3.1, the NSG P is not simple.*

2. *The NSG Q defined in Table 10 is a simple NSG as it has only a neutrosophic normal soft subgroup ϕ_{S_3}.*

Theorem 3.6. *Every simple NSG is non-abelian unless all the neutrosophic soft elements of the group are identical.*

Proof. Let P be a simple NSG defined over (G, E) whose atleast two neutrosophic soft elements are non-identical. If all the neutrosophic soft elements of P are identical, then P is abelian whatever G is (abelian / non-abelian). For contrary, suppose P is abelian. Then G is abelian by Remark 3.2. Let $M(\neq \phi_G)$ be a neutrosophic soft subgroup of P over (G, E). Then M is an abelian NSG defined over (G, E) and so is a normal neutrosophic soft subgroup of P by Remark 3.3. Thus P being a simple NSG has a normal neutrosophic soft subgroup $M(\neq \phi_G)$. This is a contradiction. Hence P is non-abelian. □

Theorem 3.7. *Every non-abelian NSG is simple.*

Proof. Let $M(\neq \phi_G)$ be a neutrosophic soft subgroup of a non-abelian NSG P defined over (G, E). Since P is a non-abelian NSG, then G is non-abelian classical group and so M is non-abelian NSG over (G, E). It implies M is non-normal, otherwise a non-abelian NSG P contains a normal neutrosophic soft subgroup $M(\neq \phi_G)$ which contradicts the Remark 3.3. Hence P is simple as M is arbitrary. □

4. Conclusion

In the present paper, the concept of NSC, N_{NSG}, neutrosophic soft quotient group, direct product of NSGs and simple NSG have been proposed in a new approach. These are illustrated with suitable examples also. Several related properties and structural characteristics are investigated. Some theorems have been established and verified by suitable examples. We extend these concepts in Nss theory context and expect further work in this setting.

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