Study of the group theory in neutrosophic soft sense

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Abstract: The purpose of this paper is to cultivate the group theory by means of neutrosophic soft sense in a different way. The concepts of neutrosophic soft coset, neutrosophic normal soft group, neutrosophic soft quotient group, direct product of neutrosophic soft groups and simple neutrosophic soft group have been presented in a new approach. These are illustrated by suitable examples. Their structural characteristics are investigated here in the parlance of group theory in classical sense. Two kinds of composition namely binary composition ‘◦’ between the elements of a classical group and neutrosophic soft composition / neutrosophic soft product ‘o’ between the neutrosophic soft elements of neutrosophic soft groups are used to practice here. Following the classical group theory, the concepts have been developed by using the neutrosophic soft composition directly.

Key words: Neutrosophic soft coset, Neutrosophic normal soft group, Neutrosophic soft quotient group, Direct product, Simple neutrosophic soft groups.

1. Introduction
Molodtsov [13] brought an opportunity to handle the uncertainty more precisely by introducing ‘Soft set theory’. Researchers in several real fields deal daily with the complexities of modeling uncertain data. There are different useful tools like probability theory, theory of fuzzy set [19], intuitionistic fuzzy set [3] etc to describe uncertainty. But the parametrization inadequacy makes all these efforts unfruitful. In that ground, soft set theory is remarkable because of it’s parametrization adequacy. Several authors [1, 2, 14, 15, 17, 18] extended the different algebraic structures over fuzzy set, intuitionistic fuzzy set and soft set.

A more generalisation of classical sets, fuzzy set, intuitionistic fuzzy set is ‘neutrosophic set’ (NS) revealed by Smarandache [16]. It is recently being practiced in development of various mathematical structures and decision making. The decision makers can get an opportunity to include their hesitation in decision making by this theory. Intuitionistic fuzzy set theory can not meet that point. Another advantage of NS theory over intuitionistic fuzzy set is that the characters representing an object are independent and appear explicitly. The combination of NS and soft set was given first by Maji [12] and thus the notion of ‘neutrosophic soft set’ (Nss) was brought to light. This concept has been practiced by several researchers [4–11] to develop different tracks of mathematics.

This paper helps to investigate the characteristics related to neutrosophic soft group in a new direction. After given some preliminary useful definitions in Section 2, the study is moved to Section 3 to state the main results. Here, the concept of neutrosophic soft coset, neutrosophic normal soft group, neutrosophic soft quotient group, direct product of neutrosophic soft groups and simple neutrosophic soft group are introduced in a new
direction following the sense of classical group theory. Finally, the Section 4 deals the present work at a glance.

2. Preliminaries

We recall some necessary definitions and results to make out the main thought.

Definition 2.1. [4] A continuous $t$-norm $*$ and $t$-conorm $\diamond$ are two continuous binary operations assigning $[0,1] \times [0,1] \to [0,1]$ and obey the under stated principles:

(i) $* \text{ and } \diamond$ are both commutative and associative.

(ii) $u * 1 = 1 * u = u$ and $u \diamond 0 = 0 \diamond u = u$, $\forall u \in [0,1]$.

(iii) $u * v \leq p * q$ and $u \diamond v \leq p \diamond q$ if $u \leq p$, $v \leq q$ with $u, v, p, q \in [0,1]$.

$u * v = uv, u \diamond v = \min\{u, v\}, u \diamond v = \max\{u + v - 1, 0\}$ are most useful $t$-norms and $u \diamond v = u + v - uv, u \diamond v = \max\{u, v\}, u \diamond v = \min\{u + v, 1\}$ are most useful $t$-conorms.

Definition 2.2. [13] A soft set on an initial universe $X$ is presented by a pair $(M, D)$ where $D \subseteq E$, the parametric set and $M$ maps $D \to \wp(X)$, the power set of $X$.

Definition 2.3. [16] An NS $Q$ on an initial universe $X$ is presented by three characterisations namely true value $T_Q$, indeterminant value $I_Q$ and false value $F_Q$ so that $T_Q, I_Q, F_Q : X \to [-0,1^+]$. Thus $Q$ can be designed as: $\{< u, (T_Q(u), I_Q(u), F_Q(u)) > \mid u \in X\}$ with $0 \leq \sup T_Q(u) + \sup I_Q(u) + \sup F_Q(u) \leq 3^+$. Here $1^+ = 1 + \delta$, where 1 is standard part and $\delta$ is non-standard part. Similarly $-0 = 0 - \delta$. The non-standard set $[-0,1^+]$ is basically practiced in philosophical ground and because of the difficulty to adopt it in real field, the standard subset of $[-0,1^+]$ i.e., $[0,1]$ is applicable in real neutrosophic environment.

Definition 2.4. [12] An Nss on an initial universe $X$ is presented by a pair $(N, B)$ where $B \subseteq E$, the parametric set and $N$ maps $B \to NS(X)$, the set of all NSs of $X$.


Definition 2.5. [11] An Nss $Q$ on $(X, E)$, $X$ being the universe set and $E$ being the parametric set, is presented by an ordered pair $(e, f_Q(e)), e \in E$ where $f_Q$ maps $E \to NS(X)$, the set of all NSs on $X$ and is given by $f_Q(e) = \{< u, (T_Q(u), I_Q(u), F_Q(u)) > \mid u \in X\}$ with $T_Q(u), I_Q(u), F_Q(u) \in [0,1]$ and $0 \leq T_Q(u) + I_Q(u) + F_Q(u) \leq 3$.

Definition 2.6. [5] Consider two Nss $P$ and $Q$ on the common universe $U$ via parametric set $E$. Then,

1. $P$ is called neutrosophic soft subset of $Q$, denoted as $P \subseteq Q$, when $T_{f_P(e)}(u) \leq T_{f_Q(e)}(u), I_{f_P(e)}(u) \geq I_{f_Q(e)}(u), F_{f_P(e)}(u) \geq F_{f_Q(e)}(u), \forall e \in E, u \in X$.

2. the ‘AND’ operation $(P \cap Q)$ is also an Nss and is defined by:

\[ M = \{ (e, e') \mid < u, T_{f_M(e, e')}(u), I_{f_M(e, e')}(u), F_{f_M(e, e')}(u) > \mid u \in X \} : (e, e') \in E \times E \]

where $T_{f_M(e, e')}(u) = T_{f_P(e)}(u) \cap T_{f_Q(e')}(u), I_{f_M(e, e')}(u) = I_{f_P(e)}(x) \cap I_{f_Q(e')}(u)$ and $F_{f_M(e, e')}(u) = F_{f_P(e)}(u) \cap F_{f_Q(e')}(u)$.

3. the ‘OR’ operation $(P \lor Q)$ is also an Nss and is defined by:

\[ K = \{ (e, e') \mid < u, T_{f_K(e, e')}(u), I_{f_K(e, e')}(u), F_{f_K(e, e')}(u) > \mid u \in X \} : (e, e') \in E \times E \]

where $T_{f_K(e, e')} (u) = T_{f_P(e)}(u) \lor T_{f_Q(e')}(u), I_{f_K(e, e')} (u) = I_{f_P(e)}(u) \lor I_{f_Q(e')}(u)$ and $F_{f_K(e, e')} (u) = F_{f_P(e)}(u) \lor F_{f_Q(e')}(u)$.
Definition 2.7. [6] An Nss function \((\psi, \xi)\) is presented by \((X, E) \rightarrow (Y, E)\) where \(\psi : X \rightarrow Y\) and \(\xi : E \rightarrow E\). Define two Nss \(P\) on \((X, E)\) and \(Q\) on \((Y, E)\). Then,

(1) the image of \(P\) under \((\psi, \xi)\) is an Nss \((\psi, \xi)(P)\) on \((Y, E)\) and it is defined as : \((\psi, \xi)(P) = \{ < \xi(a), f_{\psi(P)}(\xi(a)) > : a \in E \}\) where \(\forall b \in \xi(E), \forall v \in Y\),

\[
T_{f_{\psi(P)}(b)}(v) = \begin{cases} 
\max_{\psi(u) = v} \max_{\xi(a) = b} [T_{f_P(a)}(u)], & \text{for } u \in \psi^{-1}(v) \\
0, & \text{otherwise.}
\end{cases}
\]

\[
I_{f_{\psi(P)}(b)}(v) = \begin{cases} 
\min_{\psi(u) = v} \min_{\xi(a) = b} [I_{f_P(a)}(u)], & \text{for } u \in \psi^{-1}(v) \\
1, & \text{otherwise.}
\end{cases}
\]

\[
F_{f_{\psi(P)}(b)}(v) = \begin{cases} 
\min_{\psi(u) = v} \min_{\xi(a) = b} [F_{f_P(a)}(u)], & \text{for } u \in \psi^{-1}(v) \\
1, & \text{otherwise.}
\end{cases}
\]

(2) the pre-image of \(Q\) under \((\psi, \xi)\), is an Nss \((\psi, \xi)^{-1}(Q)\) on \((X, E)\) and it is defined as, \(\forall u \in \xi^{-1}(E), \forall v \in X\),

\[
T_{f_{\psi^{-1}(Q)}(u)}(v) = T_{f_Q(I_{\xi}(\psi(u)))}(v), \
I_{f_{\psi^{-1}(Q)}(u)}(v) = I_{f_Q(I_{\xi}(\psi(u)))}(v), \
F_{f_{\psi^{-1}(Q)}(u)}(v) = F_{f_Q(I_{\xi}(\psi(u)))}(v).
\]

\((\psi, \xi)\) is injective (surjective) when \(\psi\) and \(\xi\) both are injective (surjective).

Definition 2.8. [4] The neutrosophic soft product of Nss \(P\) and \(Q\) defined on a groupoid \(G\) is denoted by \(PoQ\) and it is also an Nss \(S\) defined as, for \((a, b) \in E \times E\) and \(u \in G\),

\[
T_{f_{(a,b)}(u)}(v) = \begin{cases} 
\max_{u = xv} [T_{f_P(a)}(x) * T_{f_Q(b)}(v)] & \text{if } u \text{ can not be put as } u = xv. \\
0 & \text{otherwise.}
\end{cases}
\]

\[
I_{f_{(a,b)}(u)}(v) = \begin{cases} 
\min_{u = xv} [I_{f_P(a)}(x) \circ I_{f_Q(b)}(v)] & \text{if } u \text{ can not be put as } u = xv. \\
1 & \text{otherwise.}
\end{cases}
\]

\[
F_{f_{(a,b)}(u)}(v) = \begin{cases} 
\min_{u = xv} [F_{f_P(a)}(x) \circ F_{f_Q(b)}(v)] & \text{if } u \text{ can not be put as } u = xv. \\
1 & \text{otherwise.}
\end{cases}
\]

Definition 2.9. [8] 1. The null Nss on \((X, E)\) is denoted by \(\phi_X\) and is defined by \((f_{\phi_X}(e))(u) = (0, 1, 1), \forall e \in E, \forall u \in X\).

2. The absolute Nss on \((X, E)\) is denoted by \(1_X\) and is defined by \((f_{1_X}(e))(u) = (1, 0, 0), \forall e \in E, \forall u \in X\).

\(\phi_X = 1_X\) and \(1_X = \phi_X\).

Definition 2.10. [8] 1. An NS \((e, f_Q(e)), e \in E\) in an Nss \(Q\) over \((X, E)\) is called a neutrosophic soft point denoted by \(e_Q\), if \(f_Q(e) \notin \phi_X\) and \(f_Q(e') \in \phi_X\ \forall e' \in E - \{e\}\).

2. The complement of \(e_Q\) is also a neutrosophic soft point \(e_Q^C\) such that \(f_Q^C(e) = (f_Q(e))^C\) hold.

3. A neutrosophic soft point \(e_Q \in P\), an Nss if \(f_Q(e) \leq f_P(e)\) for \(e \in E\).

Definition 2.11. [5] An NS \((G, \circ)\) defined on a crisp group \((G, \circ)\) is called a neutrosophic subgroup of \((G, \circ)\) with respect to the following sets of condition.

\[
(i) \quad \begin{cases} 
T_Q(u \circ v) \geq T_Q(u) * T_Q(v) \\
I_Q(u \circ v) \leq I_Q(u) \circ I_Q(v) \\
F_Q(u \circ v) \leq F_Q(u) \circ F_Q(v), \forall u, v \in G.
\end{cases}
\]

\[
(ii) \quad \begin{cases} 
T_Q(u^{-1}) \geq T_Q(u) \\
I_Q(u^{-1}) \leq I_Q(u) \\
F_Q(u^{-1}) \leq F_Q(u), \forall u \in G.
\end{cases}
\]

An Nss \(Q\) on \(((G, \circ), E)\) will be a neutrosophic soft group (NSG) if \(f_Q(e)\) is a neutrosophic subgroup of \((G, \circ)\), \(\forall e \in E\).

Over \(((G, \circ), E)\), an NSG \(P\) is called a neutrosophic soft subgroup of another NSG \(Q\) if \(P \subseteq Q\).
3. Main Result

Earlier, we have defined the left and right neutrosophic soft coset (NSC) of an Nss $P$ over a classical group $G$ in the paper [6]. Here we shall extend this concept for a neutrosophic soft subgroup $M$ of an NSG $P$. Through out the study, unless otherwise stated we shall treat $G$ as a classical group and $E$ as a parametric set.

**Definition 3.1.** Over $((G, \circ), E)$, define two NSGs $M, P$ with $M \subseteq P$ and $e_1, e_2 \in E$. Then for a fixed but arbitrary neutrosophic soft element $f_P(e_1) \in P$, the left NSC of $M$ in $P$ is:

$$f_P(e_1) o M = \{ f_P(e_1) o f_M(e_2) : f_M(e_2) \in M \}$$

$$= \{ < u, (T_{f_L(e_1,e_2)}(u), I_{f_L(e_1,e_2)}(u), F_{f_L(e_1,e_2)}(u)) > \in G : f_M(e_2) \in M \}$$

for $f_L(e_1, e_2) = f_P(e_1) o f_M(e_2)$

Similarly, for $f_P(e_1) \in P$, the right NSC of $M$ in $P$ is:

$$M o f_P(e_1) = \{ f_M(e_2) o f_P(e_1) : f_M(e_2) \in M \}$$

$$= \{ < u, (T_{f_Q(e_2,e_1)}(u), I_{f_Q(e_2,e_1)}(u), F_{f_Q(e_2,e_1)}(u)) > \in G : f_M(e_2) \in M \}$$

for $f_Q(e_2, e_1) = f_M(e_2) o f_P(e_1)$

**Example 3.1.** Let us consider two NSGs $M, P$ over $(G, E)$ given in Table 1 and Table 2, respectively with respect to $t$-norm $u \ast v = uv$ and $s$-norm $u \circ v = u + v - uv$ where $G = (\{1, \omega, \omega^2\}, \cdot)$ is the multiplicative group of cube root of unity and $E = \{a, b, c\}$.

<table>
<thead>
<tr>
<th>Table 1. Table for NSG $M$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_M(a)$</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>$\omega$</td>
</tr>
<tr>
<td>$\omega^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2. Table for NSG $P$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_P(a)$</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>$\omega$</td>
</tr>
<tr>
<td>$\omega^2$</td>
</tr>
</tbody>
</table>

Clearly, $M$ is a neutrosophic soft subgroup of $P$. The left NSC of $M$ in $P$ is $\{f_P(a) o M, f_P(b) o M, f_P(c) o M\}$ and it is given by Table 3.

For convenience of Table 3, the result of $\omega^2$ in $f_P(a) o f_M(b)$ is provided. $u \ast v = \min\{u, v\}$ and
Similarly, the right NSC of \( M \) in \( P \) is \( \{ M_{f_P(a)}, M_{f_P(b)}, M_{f_P(c)} \} \) and is given by Table 4.

**Remark 3.1.** From the above example we see that the nature of coset in an NSG is dissimilar to that in a classical group. We take the case of left NSC only. Similar conclusion holds in case of right NSC also.

(i) Any pair of NSCs \( f_P(a)\omega M, f_P(b)\omega M \) and \( f_P(c)\omega M \) are neither equal nor disjoint. They have only one common element \( f_P(a)\omega f_M(b) = f_P(b)\omega f_M(b) = f_P(c)\omega f_M(b) = \{ < 1, (0.4,0.6,0.6) >, < \omega, (0.4,0.6,0.6) >, < \omega^2, (0.4,0.6,0.6) > \} \).
For the left NSC of $M$ in $P$, each element $f_P(a)_{oM}(a)$, $f_P(b)_{oM}(a)$, $f_P(c)_{oM}(a)$, $\cdots \in P \wedge P$
by Definition 2.10, as $f_P(a)_{oM}(a) \subseteq f_{P\wedge P}(a,a)$, $f_P(b)_{oM}(a) \subseteq f_{P\wedge P}(b,a)$ and so on. But the cosets
do not make partition of $P \wedge P$. Here we use the term ‘partition’ in the sense that $\cup_v (f_P(v)_{oM}) = P \wedge P$ (i.e., in a particular case it may happen in the Example 3.1 that $f_P(a)_{oM} = f_{P\wedge P}(a,a), f_P(b)_{oM} = f_{P\wedge P}(b,b), f_P(c)_{oM} = f_{P\wedge P}(c,c)$, $\forall f_M(e) \in M$ and so on) and $\forall a, b(a \neq b) \in E, (f_P(a)_{oM}) \cap (f_P(b)_{oM})$
is not identical with an element of $P \wedge P$. We do not use the expression $(f_P(a)_{oM}) \cap (f_P(b)_{oM}) \notin P \wedge P$ due
to the Definition 2.10.

That is why there is a problem to develop the Lagrange theorem for NSG. So we have introduced here
only the concept of Lagrange NSG.

**Definition 3.2.** Let $P$ be an NSG over $(G, E)$. The number of distinct elements in $P$ is called the order of $P$
and is denoted by $|P|$. A finite NSG $P$ over $(G, E)$ contains finite number of elements, otherwise it is called
infinite NSG.

**Example 3.2.** 1. Let us consider an NSG $P$ over $(V, E)$ as given in Table 5 where $V = \{e, a, b, c\}$ be the Klein’s
4 group and $E = \{\alpha, \beta, \gamma, \delta\}$ be the set of parameters. $\ast$ and $\circ$ are $u \ast v = \max\{u + v - 1, 0\}, u \circ v = \min\{u + v, 1\}$.

<table>
<thead>
<tr>
<th>Table 5. Table for NSG $P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_P(\alpha)$</td>
</tr>
<tr>
<td>$e$ (0.6, 0.3, 0.1)</td>
</tr>
<tr>
<td>$a$ (0.7, 0.2, 0.7)</td>
</tr>
<tr>
<td>$b$ (0.7, 0.2, 0.5)</td>
</tr>
<tr>
<td>$c$ (0.6, 0.3, 0.2)</td>
</tr>
</tbody>
</table>

It is a finite NSG.

2. Consider another NSG $M$ over $(G, E)$ where $E = \mathbb{N}$ (the set of natural numbers), be the parametric set
and $G = (\mathbb{Z}, +)$ be the group of all integers. Define a mapping $f_M : \mathbb{N} \rightarrow NS(\mathbb{Z})$ where, for any $n \in \mathbb{N}$ and
$x \in \mathbb{Z}$,

$$T_{f_M(n)}(x) = \begin{cases} 0, & x \text{ odd} \\ \frac{1}{n+1}, & x \text{ even} \end{cases} \quad I_{f_M(n)}(x) = \begin{cases} \frac{1}{n+1}, & x \text{ odd} \\ 0, & x \text{ even} \end{cases} \quad F_{f_M(n)}(x) = \begin{cases} 1 - \frac{1}{n}, & x \text{ odd} \\ 0, & x \text{ even} \end{cases}$$

Corresponding $t$-norm and $s$-norm are taken as $u \ast v = \min\{u, v\}, u \circ v = \max\{u, v\}$. It is an infinite NSG.

**Definition 3.3.** Let $P, M$ be two finite NSGs over $(G, E)$ such that $P \subset M$. If $|P|/|M|$, then $P$ is called a
Lagrange neutrosophic soft subgroup. For a finite NSG $M$, if all it’s neutrosophic soft subgroups are Lagrange
then $M$ is said to be a Lagrange NSG.

An NSG $M$, having no Lagrange neutrosophic soft subgroup, is called Lagrange free NSG.

**Example 3.3.** We consider the NSG $M$ over $([\mathbb{Z}_3, +], E)$ given in Table 6, where $E = \{e_1, e_2, e_3, e_4\}$.
$u \ast v = \max\{u + v - 1, 0\}, u \circ v = \min\{u + v, 1\}$ are corresponding $t$-norm and $s$-norm.

We construct two neutrosophic soft subgroups $P, Q$ of $M$ over that $([\mathbb{Z}_3, +], E)$ given in Table 7 and
Table 8, respectively.

Here $|P| = 4$ and $|Q| = 3$, as $f_Q(e_2) = f_Q(e_3)$. Thus $|P|/|M|$ but $|Q|$ does not divide $|M|$ i.e., the
order of each neutrosophic soft subgroup of $M$ does not divide $|M|$. So in general, an NSG does not satisfy the
Lagrange theorem in classical sense. Moreover $P$ is called a Lagrange neutrosophic soft subgroup of $M$. 


Table 6. Table for NSG $M$

<table>
<thead>
<tr>
<th></th>
<th>$f_M(e_1)$</th>
<th>$f_M(e_2)$</th>
<th>$f_M(e_3)$</th>
<th>$f_M(e_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>(0.7,0.4,0.3)</td>
<td>(0.6,0.5,0.6)</td>
<td>(0.3,0.7,0.4)</td>
<td>(0.4,0.5,0.5)</td>
</tr>
<tr>
<td>$\top$</td>
<td>(0.5,0.7,0.2)</td>
<td>(0.7,0.3,0.4)</td>
<td>(0.6,0.5,0.3)</td>
<td>(0.1,0.7,0.6)</td>
</tr>
<tr>
<td>$\mathbb{0}$</td>
<td>(0.4,0.8,0.5)</td>
<td>(0.5,0.6,0.7)</td>
<td>(0.5,0.4,0.2)</td>
<td>(0.5,0.8,0.4)</td>
</tr>
</tbody>
</table>

Table 7. Tabular form of neutrosophic soft subgroup $P$

<table>
<thead>
<tr>
<th></th>
<th>$f_P(e_1)$</th>
<th>$f_P(e_2)$</th>
<th>$f_P(e_3)$</th>
<th>$f_P(e_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>(0.6,0.5,0.4)</td>
<td>(0.5,0.6,0.7)</td>
<td>(0.2,0.8,0.5)</td>
<td>(0.3,0.6,0.6)</td>
</tr>
<tr>
<td>$\top$</td>
<td>(0.4,0.8,0.3)</td>
<td>(0.6,0.4,0.5)</td>
<td>(0.5,0.6,0.4)</td>
<td>(0.1,0.8,0.7)</td>
</tr>
<tr>
<td>$\mathbb{0}$</td>
<td>(0.3,0.9,0.6)</td>
<td>(0.4,0.7,0.8)</td>
<td>(0.4,0.5,0.3)</td>
<td>(0.4,0.9,0.5)</td>
</tr>
</tbody>
</table>

Table 8. Tabular form of neutrosophic soft subgroup $Q$

<table>
<thead>
<tr>
<th></th>
<th>$f_Q(e_1)$</th>
<th>$f_Q(e_2)$</th>
<th>$f_Q(e_3)$</th>
<th>$f_Q(e_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>(0.5,0.6,0.5)</td>
<td>(0.3,0.7,0.8)</td>
<td>(0.3,0.7,0.8)</td>
<td>(0.4,0.7,0.6)</td>
</tr>
<tr>
<td>$\top$</td>
<td>(0.3,0.7,0.4)</td>
<td>(0.5,0.5,0.6)</td>
<td>(0.5,0.5,0.6)</td>
<td>(0.1,0.7,0.8)</td>
</tr>
<tr>
<td>$\mathbb{0}$</td>
<td>(0.1,0.8,0.7)</td>
<td>(0.2,0.7,0.7)</td>
<td>(0.2,0.7,0.7)</td>
<td>(0.3,0.8,0.4)</td>
</tr>
</tbody>
</table>

**Theorem 3.1.** Let $M$, $P$ be two NSGs over $(G,E)$ such that $M \subseteq P$. Then any two left (right) NSCs of $M$ in $P$ have same cardinality.

**Proof.** Let $f_P(a)oM$ and $f_P(b)oM$ be two left NSCs of $M$ in $P$ over $(G,E)$. We define a neutrosophic soft mapping $(\psi, \xi): f_P(a)oM \rightarrow f_P(b)oM$ by $(\psi, \xi)(f_P(a)mf_M(e)) = f_P(b)mf_M(e'), \forall f_M(e), f_M(e') \in M$, i.e., $(\psi, \xi)(f_L(a,e)) = f_L(b,e')$ where $f_P(a)mf_M(e) = f_L(a,e)$. We are to show that $(\psi, \xi)$ is bijective. Let,

$$
(\psi, \xi)(f_L(a,e_1)) = (\psi, \xi)(f_L(a,e_2))
$$

$\Rightarrow$ $T_{f_L(x)}(\xi(a,e_1)) = T_{f_L(x)}(\xi(a,e_2))$, $I_{f_L(x)}(\xi(a,e_1)) = I_{f_L(x)}(\xi(a,e_2))$,

$F_{f_L(x)}(\xi(a,e_1)) = F_{f_L(x)}(\xi(a,e_2)), \forall y \in G$.

$\Rightarrow$ $\max_{\psi(x)} \max_{\xi(a,e_1)} [T_{f_L(x)}(\xi(a,e_1))] = \max_{\psi(x)} \max_{\xi(a,e_2)} [T_{f_L(x)}(\xi(a,e_2))],$

$\min_{\psi(x)} \min_{\xi(a,e_1)} [I_{f_L(x)}(\xi(a,e_1))] = \min_{\psi(x)} \min_{\xi(a,e_2)} [I_{f_L(x)}(\xi(a,e_2))],$

$\min_{\psi(x)} \min_{\xi(a,e_1)} [F_{f_L(x)}(\xi(a,e_1))] = \min_{\psi(x)} \min_{\xi(a,e_2)} [F_{f_L(x)}(\xi(a,e_2))]$, if $x \in \psi^{-1}(y)$

and if $x \notin \psi^{-1}(y)$, the equality is also obvious from definition.

$\Rightarrow$ $T_{f_L(x)}(\xi(a,e_1)) = T_{f_L(x)}(\xi(a,e_2)), \forall x \in G$ (as $x$ is arbitrary).

$\Rightarrow$ $f_L(a,e_1) = f_L(a,e_2)$

Thus $(\psi, \xi)$ is injective and from formation $(\psi, \xi)$ is onto also. This ends the proof.

The theorem can be verified from the Example 3.1.

**Definition 3.4.** An NSG $P$ over the group $G$ is called abelian NSG if $f_P(a)mf_P(b) = f_P(b)mf_P(a), \forall a, b \in E$, otherwise it is non-abelian.
Example 3.4. 1. The NSG $P$ defined in Example 3.1 is abelian.

2. We define another NSG $Q$ over $(S_3, E)$ where $S_3$ is the group (Table 9) of all permutations on the set $S = \{1, 2, 3\}$ i.e., $S_3 = \{\rho_0(i), \rho_1(123), \rho_2(132), \rho_3(23), \rho_4(13), \rho_5(12)\}$ and $E = \{a, b\}$, as given in Table 10.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\rho_0$</th>
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<tbody>
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<td>$\rho_2$</td>
<td>$\rho_0$</td>
</tr>
</tbody>
</table>

$u \star v = \max\{u + v - 1, 0\}, u \circ v = \min\{u + v, 1\}$ are $t$-norm and $s$-norm. Then $Q$ is non-abelian NSG over $(S_3, E)$.

To verify, we estimate the truth membership functions of $\rho_1$ in $F_Q(a) \star F_Q(b)$ and $F_Q(b) \star F_Q(a)$ with respect to the said $t$-norm. Here $\rho_1 = \rho_0 \cdot \rho_1 = \rho_1 \cdot \rho_0 = \rho_2 \cdot \rho_2 = \rho_3 \cdot \rho_4 = \rho_4 \cdot \rho_5 = \rho_5 \cdot \rho_3$. Now,

$T_{FL(a,b)}(\rho_1) = \max\{T_{F_Q(a)}(\rho_0) * T_{F_Q(b)}(\rho_1), T_{F_Q(a)}(\rho_1) * T_{F_Q(b)}(\rho_0), T_{F_Q(a)}(\rho_2) * T_{F_Q(b)}(\rho_2), T_{F_Q(a)}(\rho_3) * T_{F_Q(b)}(\rho_4), T_{F_Q(a)}(\rho_4) * T_{F_Q(b)}(\rho_3), T_{F_Q(a)}(\rho_5) * T_{F_Q(b)}(\rho_5), T_{F_Q(a)}(\rho_5) * T_{F_Q(b)}(\rho_3)\}$

$= \max\{0.4 * 0.5, 0.6 * 0.6, 0.4 * 0.5, 0.5 * 0.2, 0.7 * 0.3, 0.4\}$

$= \max\{0.4, 0.6, 0.4, 0.3, 0.4\} = 0.4$

$T_{FL(b,a)}(\rho_1) = \max\{T_{F_Q(b)}(\rho_0) * T_{F_Q(a)}(\rho_1), T_{F_Q(b)}(\rho_1) * T_{F_Q(a)}(\rho_0), T_{F_Q(b)}(\rho_2) * T_{F_Q(a)}(\rho_2), T_{F_Q(b)}(\rho_3) * T_{F_Q(a)}(\rho_4), T_{F_Q(b)}(\rho_4) * T_{F_Q(a)}(\rho_3), T_{F_Q(b)}(\rho_5) * T_{F_Q(a)}(\rho_5), T_{F_Q(b)}(\rho_5) * T_{F_Q(a)}(\rho_3)\}$

$= \max\{0.6 * 0.6, 0.5 * 0.4, 0.4 * 0.4, 0.7 * 0.2, 0.3 * 0.7, 0.5\}$

$= \max\{0.4, 0.6, 0.4, 0.3, 0.4\} = 0.4$

Remark 3.2. An NSG $P$ will be abelian or non-abelian according as the classical group $G$ and the parametric set $E$ over which $P$ is defined are together abelian or non-abelian, respectively unless all neutrosophic soft elements in $P$ are identical.

Definition 3.5. A neutrosophic soft subgroup $M$ of an NSG $P$ over $(G, E)$ is called neutrosophic normal soft subgroup if $f_P(e) \circ M = M \circ f_P(e), \forall f_P(e) \in P$. 

8
Example 3.5. 1. From Table 3 and Table 4, it is clear that $M$ is a neutrosophic normal soft subgroup of $P$ over $(G,E)$.

2. The null NSG $\phi_G$ over $(G,E)$ is a neutrosophic normal soft subgroup of every NSG $P$ defined over same $(G,E)$.

3. The NSG $Q$ in Table 11 has no neutrosophic normal soft subgroup except $\phi_{S_3}$.

Remark 3.3. (i) For a neutrosophic normal soft subgroup $M$ of an NSG $P$, each left coset and right coset of $M$ in $P$ are equal. We then call only neutrosophic soft coset of $M$ in $P$ instead of left and right coset separately.

(ii) Every neutrosophic soft subgroup of an abelian NSG is always normal as well as abelian. In particular, each abelian NSG is itself normal.

(iii) Every non-null neutrosophic soft subgroup of a non-abelian NSG is non-normal.

(iv) Every neutrosophic normal soft subgroup of an NSG is abelian. In particular, each neutrosophic normal soft group $(N_{NSG})$ is itself abelian.

Theorem 3.2. Let $P$ be an $N_{NSG}$ over $(X,E)$ and $(\psi,\xi) : (X,E) \rightarrow (Y,E)$ be a neutrosophic soft epimorphism where $X,Y$ are two classical groups and $E$ is a parametric set. Then $(\psi,\xi)(P)$ is an $N_{NSG}$ over $(Y,E)$.

Proof. Here $P$ is abelian NSG over $(X,E)$ and so both $X,E$ are abelian together. For $x,x_1,x_2 \in X$ with $x = x_1 \circ x_2$ and $a,b \in E$, we have,

$$f_P(a) \circ P = P \circ f_P(a), \forall f_P(a) \in P$$

$$\Rightarrow f_P(a) \circ f_P(b) = f_P(b) \circ f_P(a), \forall f_P(a), f_P(b) \in P$$

$$\Rightarrow \max_{x=x_1 \circ x_2} [T_{f_P(a)}(x_1) \ast T_{f_P(b)}(x_2)] = \max_{x=x_1 \circ x_2} [T_{f_P(b)}(x_1) \ast T_{f_P(a)}(x_2)],$$

$$\min_{x=x_1 \circ x_2} [I_{f_P(a)}(x_1) \circ I_{f_P(b)}(x_2)] = \min_{x=x_1 \circ x_2} [I_{f_P(b)}(x_1) \circ I_{f_P(a)}(x_2)],$$

$$\min_{x=x_1 \circ x_2} [F_{f_P(a)}(x_1) \circ F_{f_P(b)}(x_2)] = \min_{x=x_1 \circ x_2} [F_{f_P(b)}(x_1) \circ F_{f_P(a)}(x_2)].$$

As $(\psi,\xi)$ is a neutrosophic soft epimorphism, for $y,y_1,y_2 \in Y$ and $a',b' \in E$ such that $\psi(x) = y, \psi(x_1) = y_1, \psi(x_2) = y_2$ and $\xi(a) = a', \xi(b) = b'$.

Also $\psi(x) = \psi(x_1 \circ x_2) = \psi(x_1) \circ \psi(x_2)$ i.e., $y = y_1 \circ y_2$. Then,

$$\max_{y=y_1 \circ y_2} [T_{\psi^{-1}}(a') \ast T_{\psi^{-1}}(b')](y_1 \ast T_{\psi^{-1}}(b')](y_2)]$$

$$= \max_{y=y_1 \circ y_2} \left\{ \max_{y_1 = \psi(x_1)} [T_{f_P(a)}(x_1)] \ast \max_{y_2 = \psi(x_2)} [T_{f_P(b)}(x_2)] \right\}$$

$$= \max_{y=y_1 \circ y_2} \left\{ \max_{\psi(x_1) \circ \psi(x_2)} [T_{f_P(a)}(x_1) \ast T_{f_P(b)}(x_2)] \right\}$$

$$= \max_{\psi(x_1) \circ \psi(x_2)} \left\{ \max_{x=x_1 \circ x_2} [T_{f_P(a)}(x_1) \ast T_{f_P(b)}(x_2)] \right\}$$

$$= \max_{\psi(x_1) \circ \psi(x_2)} \left\{ \max_{x=x_1 \circ x_2} [T_{f_P(b)}(x_1) \ast T_{f_P(a)}(x_2)] \right\}$$

9
by the condition of truth membership function provided

\[
= \max_{\psi(x_1, x_2)} \max_{\xi(b, a)} \left\{ \max_{x=x_1 \# x_2} \left[ T_{f_{\psi(b)}(x_1)} \ast T_{f_{\psi(a)}(x_2)} \right] \right\}
\]

( as \( E \) is abelian in classical sense)

\[
= \max_{y=y_1 \# y_2} \left\{ \max_{\psi(x_1, x_2)} \max_{\xi(b, a)} \left[ T_{f_{\psi(b)}(x_1)} \ast T_{f_{\psi(a)}(x_2)} \right] \right\}
\]

\[
= \max_{y=y_1 \# y_2} \left\{ \max_{\psi(x_1, x_2)} \max_{\xi(b, a)} \left[ T_{f_{\psi(b)}(x_1)} \ast T_{f_{\psi(a)}(x_2)} \right] \right\}
\]

\[
= \max_{y=y_1 \# y_2} \left\{ \max_{\psi(x_1, x_2)} \max_{\xi(b, a)} \left[ T_{f_{\psi(b)}(x_1)} \ast T_{f_{\psi(a)}(x_2)} \right] \right\}
\]

\[
= \max_{y=y_1 \# y_2} \left\{ \max_{\psi(x_1, x_2)} \max_{\xi(b, a)} \left[ T_{f_{\psi(b)}(x_1)} \ast T_{f_{\psi(a)}(x_2)} \right] \right\}
\]

(1)

Next,

\[
\min_{y=y_1 \# y_2} \left[ I_{f_{\psi(b')}(a')}(y_1) \diamond I_{f_{\psi(b')}(a')}(y_2) \right]
\]

\[
= \min_{y=y_1 \# y_2} \left\{ \min_{\psi(x_1 \# x_2)} \min_{\xi(b, a)} \left[ I_{f_{\psi(b)}(x_1)} \diamond I_{f_{\psi(a)}(x_2)} \right] \right\}
\]

\[
= \min_{y=y_1 \# y_2} \left\{ \min_{\psi(x_1 \# x_2)} \min_{\xi(b, a)} \left[ I_{f_{\psi(b)}(x_1)} \diamond I_{f_{\psi(a)}(x_2)} \right] \right\}
\]

\[
= \min_{y=y_1 \# y_2} \left\{ \min_{\psi(x_1 \# x_2)} \min_{\xi(b, a)} \left[ I_{f_{\psi(b)}(x_1)} \diamond I_{f_{\psi(a)}(x_2)} \right] \right\}
\]

\[
= \min_{y=y_1 \# y_2} \left\{ \min_{\psi(x_1 \# x_2)} \min_{\xi(b, a)} \left[ I_{f_{\psi(b)}(x_1)} \diamond I_{f_{\psi(a)}(x_2)} \right] \right\}
\]

\[
= \min_{y=y_1 \# y_2} \left\{ \min_{\psi(x_1 \# x_2)} \min_{\xi(b, a)} \left[ I_{f_{\psi(b)}(x_1)} \diamond I_{f_{\psi(a)}(x_2)} \right] \right\}
\]

\[
= \min_{y=y_1 \# y_2} \left\{ \min_{\psi(x_1 \# x_2)} \min_{\xi(b, a)} \left[ I_{f_{\psi(b)}(x_1)} \diamond I_{f_{\psi(a)}(x_2)} \right] \right\}
\]

(2)

Finally,

\[
\min_{y=y_1 \# y_2} \left[ F_{f_{\psi(b')}(a')}(y_1) \diamond F_{f_{\psi(b')}(a')}(y_2) \right]
\]

\[
= \min_{y=y_1 \# y_2} \left\{ \min_{\psi(x_1 \# x_2)} \min_{\xi(b, a)} \left[ F_{f_{\psi(b)}(x_1)} \diamond F_{f_{\psi(b)}(x_2)} \right] \right\}
\]

\[
= \min_{y=y_1 \# y_2} \left\{ \min_{\psi(x_1 \# x_2)} \min_{\xi(b, a)} \left[ F_{f_{\psi(b)}(x_1)} \diamond F_{f_{\psi(b)}(x_2)} \right] \right\}
\]

\[
= \min_{y=y_1 \# y_2} \left\{ \min_{\psi(x_1 \# x_2)} \min_{\xi(b, a)} \left[ F_{f_{\psi(b)}(x_1)} \diamond F_{f_{\psi(b)}(x_2)} \right] \right\}
\]

\[
= \min_{y=y_1 \# y_2} \left\{ \min_{\psi(x_1 \# x_2)} \min_{\xi(b, a)} \left[ F_{f_{\psi(b)}(x_1)} \diamond F_{f_{\psi(b)}(x_2)} \right] \right\}
\]
From (1), (2) and (3), we have, 
\[
\begin{align*}
&= \min_{y=y_1 \circ y_2} \left\{ \min_{\psi(x_1 \circ x_2)} \min_{\xi(\text{bool})} \left[ F_{f \circ b}(x_1) \circ F_{f \circ a}(x_2) \right] \right\} \\
&= \min_{y=y_1 \circ y_2} \left\{ \min_{\psi(x_1 \circ x_2)} \left[ F_{f \circ b}(x_1) \circ F_{f \circ a}(x_2) \right] \right\} \\
&= \min_{y=y_1 \circ y_2} \left\{ \min_{\psi(x_1 \circ x_2)} \left[ F_{f \circ b}(x_1) \right] \circ \min_{y_2=\psi(x_2)} \left[ F_{f \circ a}(x_2) \right] \right\} \\
&= \min_{y=y_1 \circ y_2} \left[ F_{f \circ b}(y_1) \circ F_{f \circ a}(y_2) \right]. \\
\end{align*}
\]

(3)

From (1), (2) and (3), we have, 
\[
f_{\psi(P)}(a') o f_{\psi(P)}(b') = f_{\psi(P)}(b') o f_{\psi(P)}(a')
\]

\[
\Rightarrow f_{\psi(P)}(a') o f_{\psi(P)}(b') = f_{\psi(P)}(b') o f_{\psi(P)}(a'), \quad \text{as } f_{\psi(P)}(b') \in (\psi, \xi)(P) \text{ arbitrary.}
\]

As \( f_{\psi(P)}(a') \in (\psi, \xi)(P) \) is an arbitrary, so \((\psi, \xi)(P)\) is an \( N_{NSG} \) over \( (Y, E) \).

**Theorem 3.3.** Let \( M \) be an \( N_{NSG} \) over \( (Y, E) \) and \((\psi, \xi) : (X, E) \rightarrow (Y, E)\) be a neutrosophic soft homomorphism where \( X, Y \) are two classical groups and \( E \) is a parametric set. Then \((\psi, \xi)^{-1}(M)\) is an \( N_{NSG} \) over \( (X, E) \).

**Proof.** Here \( M \) is abelian \( \text{NSG} \) over \( (Y, E) \) and so both \( Y, E \) are abelian together. Let \( a, b \in \xi^{-1}(E) \) and \( x, x_1, x_2 \in X \) with \( x = x_1 \circ x_2 \). As \((\psi, \xi)\) is a neutrosophic soft homomorphism, so \( \psi(x) = \psi(x_1 \circ x_2) = \psi(x_1) \circ \psi(x_2) \).

Then, 
\[
f_M(\xi(a))oM = M o f_M(\xi(a)), \quad \forall f_M(\xi(a)) \in M
\]

\[
\Rightarrow f_M(\xi(a))o f_M(\xi(b)) = f_M(\xi(b)) o f_M(\xi(a)), \quad \forall f_M(\xi(a)), f_M(\xi(b)) \in M
\]

\[
\Rightarrow \max_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [T_{f_M}[\xi(a)](\psi(x_1)) \circ T_{f_M}[\xi(b)](\psi(x_2))]
\]

\[
= \max_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [T_{f_M}[\xi(b)](\psi(x_1)) \circ T_{f_M}[\xi(a)](\psi(x_2))],
\]

\[
\min_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [I_{f_M}[\xi(a)](\psi(x_1) \circ \psi(x_2))]
\]

\[
= \min_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [I_{f_M}[\xi(a)](\psi(x_1)) \circ I_{f_M}[\xi(a)](\psi(x_2))],
\]

\[
\min_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [F_{f_M}[\xi(a)](\psi(x_1)) \circ F_{f_M}[\xi(b)](\psi(x_2))]
\]

As \((\psi, \xi)\) is a neutrosophic soft homomorphism, so \( \psi(x) = \psi(x_1 \circ x_2) = \psi(x_1) \circ \psi(x_2) \).

Now, 
\[
\max_{x = x_1 \circ x_2} [T_{f -1(M)}[a](x_1) \circ T_{f -1(M)}[b](x_2)]
\]

\[
= \max_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [T_{f_M}[\xi(a)](\psi(x_1)) \circ T_{f_M}[\xi(b)](\psi(x_2))]
\]

\[
= \max_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [T_{f_M}[\xi(b)](\psi(x_1)) \circ T_{f_M}[\xi(a)](\psi(x_2))]
\]

(by the condition of truth membership function provided)

\[
= \max_{x = x_1 \circ x_2} [T_{f -1(M)}[a](x_1) \circ T_{f -1(M)}[b](x_2)]. \quad (4)
\]
Next, \[ \min_{x = x_1 \circ x_2} [I_{f_{\psi^{-1}(M)}(a)}(x_1) \circ I_{f_{\psi^{-1}(M)}(b)}(x_2)] \]
\[ = \min_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [I_{f_{M}[\xi(a)]}(\psi(x_1)) \circ I_{f_{M}[\xi(b)]}(\psi(x_2))] \]
\[ = \min_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [I_{f_{M}[\xi(b)]}(\psi(x_1)) \circ I_{f_{M}[\xi(a)]}(\psi(x_2))] \]
( by the condition of indeterminacy membership function provided ) \[ = \min_{x = x_1 \circ x_2} [I_{f_{\psi^{-1}(M)}(b)}(x_1) \circ I_{f_{\psi^{-1}(M)}(a)}(x_2)]. \] (5)

Finally, \[ \min_{x = x_1 \circ x_2} [F_{f_{\psi^{-1}(M)}(a)}(x_1) \circ F_{f_{\psi^{-1}(M)}(b)}(x_2)] \]
\[ = \min_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [F_{f_{M}[\xi(a)]}(\psi(x_1)) \circ F_{f_{M}[\xi(b)]}(\psi(x_2))] \]
\[ = \min_{\psi(x) = \psi(x_1) \circ \psi(x_2)} [F_{f_{M}[\xi(b)]}(\psi(x_1)) \circ F_{f_{M}[\xi(a)]}(\psi(x_2))] \]
\[ = \min_{x = x_1 \circ x_2} [F_{f_{\psi^{-1}(M)}(b)}(x_1) \circ F_{f_{\psi^{-1}(M)}(a)}(x_2)]. \] (6)

From (4), (5) and (6), we have,
\[ f_{\psi^{-1}(M)}(a) \circ f_{\psi^{-1}(M)}(b) = f_{\psi^{-1}(M)}(b) \circ f_{\psi^{-1}(M)}(a). \]
\[ \Rightarrow f_{\psi^{-1}(M)}(a) \circ (\psi, \xi)^{-1}(M) = (\psi, \xi)^{-1}(M) \circ f_{\psi^{-1}(M)}(a). \] [as \( f_{\psi^{-1}(M)}(b) \in (\psi, \xi)^{-1}(M) \) arbitrary.]
As \( f_{\psi^{-1}(M)}(a) \in (\psi, \xi)^{-1}(M) \) is an arbitrary, so \( (\psi, \xi)^{-1}(M) \) is an \( N_{NSG} \) over \( (X, E) \).

**Theorem 3.4.** Let \( M \) be a neutrosophic normal soft subgroup of an \( NSG \ P \) over \( (G, E) \) and \( \Omega \) be a collection of all distinct NSCs of \( M \) in \( P \). Then \( \Omega \) forms an \( NSG \) over \( (G, E \times E) \).

**Proof.** As \( M \) is an \( N_{NSG} \) defined over \( (G, E) \), then \( M \) is abelian and so \( G \) is abelian by Remark 3.2. Moreover there is no distinction between left and right NSC of \( M \) in \( P \) over \( (G, E \times E) \). Let \( (a, b) \in E \times E \) and \( x, y, z, y_1, y_2, z_1, z_2 \in G \) be arbitrary such that \( x = y \circ z, y = y_1 \circ y_2, z = z_1 \circ z_2 \). Suppose \( PoM = L \). Now,
\[ T_{f_L(a, b)}(y) * T_{f_L(a, b)}(z) \]
\[ = \max_{y = y_1 \circ y_2} [T_{f_{P}(a)}(y_1) * T_{f_{M}(b)}(y_2)] * \max_{z = z_1 \circ z_2} [T_{f_{P}(a)}(z_1) * T_{f_{M}(b)}(z_2)] \]
\[ = \max_{x = (y_1 \circ y_2) \circ (z_1 \circ z_2)} [(T_{f_{P}(a)}(y_1) * T_{f_{M}(b)}(y_2)) * (T_{f_{P}(a)}(z_1) * T_{f_{M}(b)}(z_2))] \]
\[ = \max_{x = (y_1 \circ y_2) \circ (z_1 \circ z_2)} [(T_{f_{P}(a)}(y_1) * T_{f_{P}(a)}(z_1)) * (T_{f_{M}(b)}(y_2) * T_{f_{M}(b)}(z_2))] \]
( as * is commutative ) \[ \leq \max_{x = (y_1 \circ y_2) \circ (z_1 \circ z_2)} [T_{f_{P}(a)}(y_1 \circ z_1) * T_{f_{M}(b)}(y_2 \circ z_2)] \]
( as \( M \) and \( P \) are two neutrosophic soft groups ) \[ = T_{f_L(a, b)}[(y_1 \circ z_1) \circ (y_2 \circ z_2)] \]
\[ = T_{f_L(a, b)}[(y_1 \circ y_2) \circ (z_1 \circ z_2)] \] ( as \( G \) is abelian ) \[ = T_{f_L(a, b)}(y \circ z). \]

Thus, \( T_{f_L(a, b)}(y \circ z) \geq T_{f_L(a, b)}(y) * T_{f_L(a, b)}(z). \) Next,
Proposition 3.1. Let \( M \) be a neutrosophic normal soft subgroup of an NSG \( P \) over \((G, E \times E)\). Then there exists a neutrosophic soft homomorphism \((\psi, \xi) : P \to P/M\) defined as \((\psi, \xi)(f_P(a)) = f_P(a)_{oM}, \forall f_P(a) \in P\) if \( u \ast v = \min\{u, v\} \) and \( u \circ v = \max\{u, v\} \).

Proof. Let \((\psi, \xi) : P \to P/M\) be defined as \((\psi, \xi)(f_P(a)) = f_P(a)_{oM}, \forall f_P(a) \in M\). We shall show \((\psi, \xi)\) a neutrosophic soft homomorphism in the sense that

\[
(\psi, \xi)[f_P(a)_{oM}(b)] = (\psi, \xi)[f_P(a)]_{o}(\psi, \xi)[f_P(b)], \forall a, b \in E.
\]

Thus \((\psi, \xi)\) is a neutrosophic soft homomorphism. So, \(F_{f_P(a)}(y \circ z) \leq F_{f_P(a)}(y) \circ F_{f_P(a)}(z)\) and this ends the proof.

This group \( \Omega \) is called neutrosophic soft quotient group of \( P \) by \( M \) over \((G, E \times E)\) and is denoted by \( P/M \).
\[ x \circ z, s_2 = y \circ z, s = t \circ z. \] Then,

\[
f_P(a) \circ_P f_P(b) = \{ < t, \max_{t \in G} [T_{f_P(a)}(x) * T_{f_P(b)}(y)], \min_{t \in G} [I_{f_P(a)}(x) \circ I_{f_P(b)}(y)], \min_{t \in G} [F_{f_P(a)}(x) \circ F_{f_P(b)}(y)] > : t \in G \}
\]

\[
(f_P(a) \circ_P f_P(b)) \circ_M e = \{ < s, \max_{s \in G} \max_{t \in G} [T_{f_P(a)}(x) * T_{f_P(b)}(y)] * T_{f_M(e)}(z)], \min_{t \in G} [\min_{s \in G} [I_{f_P(a)}(x) \circ I_{f_P(b)}(y)] \circ I_{f_M(e)}(z)], \min_{t \in G} [\min_{s \in G} [F_{f_P(a)}(x) \circ F_{f_P(b)}(y)] \circ F_{f_M(e)}(z)] > : s \in G \}
\]

\[
f_P(a) \circ_M e = \{ < s_1, \max_{x \in G} [T_{f_P(a)}(x) * T_{f_M(e)}(z)], \min_{x \in G} [I_{f_P(a)}(x) \circ I_{f_M(e)}(z)], \min_{s \in G} [F_{f_P(a)}(x) \circ F_{f_M(e)}(z)] > : s_1 \in G \}
\]

\[
f_P(b) \circ_M e = \{ < s_2, \max_{y \in G} [T_{f_P(b)}(y) * T_{f_M(e)}(z)], \min_{y \in G} [I_{f_P(b)}(y) \circ I_{f_M(e)}(z)], \min_{s \in G} [F_{f_P(b)}(y) \circ F_{f_M(e)}(z)] > : s_2 \in G \}
\]

Now,

\[
\max_{s_1, s_2} \max_{s_1 \in G} [T_{f_P(a)}(x) * T_{f_M(e)}(z)] \max_{s_2 \in G} [T_{f_P(b)}(y) * T_{f_M(e)}(z)]
\]

\[
= \max_{s_1, s_2} \max_{e \in G} [T_{f_P(a)}(x) * T_{f_M(e)}(z)] \max_{s \in G} [T_{f_P(b)}(y) * T_{f_M(e)}(z)]
\]

\[
= \max_{s_1, s_2} \max_{x,y \in G} [(T_{f_P(a)}(x) * T_{f_M(e)}(z)) \max_{s \in G} [(T_{f_P(b)}(y)) * T_{f_M(e)}(z)]]
\]

\[
(\text{as * and G both are commutative})
\]

\[
= \max_{s} \max_{t \in G} [T_{f_P(a)}(x) * T_{f_P(b)}(y)] * T_{f_M(e)}(z)
\]

Finally,

\[
= \min_{s} \min_{t \in G} [I_{f_P(a)}(x) \circ I_{f_M(e)}(z)] \max_{s \in G} [F_{f_P(b)}(y) \circ F_{f_M(e)}(z)]
\]

\[
= \min_{s} \min_{t \in G} [F_{f_P(a)}(x) * F_{f_M(e)}(z)] \max_{s \in G} [F_{f_P(b)}(y) \circ F_{f_M(e)}(z)]
\]

From (7), (8) and (9), we see that

\[
(f_P(a) \circ_M e) \circ (f_P(b) \circ_M e) = (f_P(a) \circ_P f_P(b)) \circ_M e
\]

and this ends the proof.
Definition 3.6. Let $M$ and $P$ be two NSGs over $(G, E)$. Their direct product is denoted by $M \otimes P$ and is defined as:

$$M \otimes P = \{ ((a,b), (f_M(a), f_P(b)) : (a,b) \in E \times E \}$$

where

$$(f_M(a), f_N(b)) = \{ (x,y), T_{f_M(a)}(x)*T_{f_P(b)}(y), I_{f_M(a)}(x) \circ I_{f_P(b)}(y), \quad F_{f_M(a)}(x) \circ F_{f_P(b)}(y) >: (x,y) \in G \times G \}$$

The definition can be extended for any finite number of NSGs.

Example 3.6. Consider NSGs $M, P$ defined in Table 1 and Table 2, respectively. Then their direct product $M \otimes P$ is given in Table 11. The $*$ and $\circ$ are taken as $u \ast v = \min\{u,v\}, u \circ v = \max\{u,v\}$.

Theorem 3.5. Let $M, P$ be two NSGs over $(G, E)$. Then their direct product $M \otimes P$ is also an NSG over $(G \times G, E \times E)$.

Proof. Let $(a, b) \in E \times E$ and $(x, y) \in G \times G$ be arbitrary such that $(x, y) = (x_1, y_1) \circ (x_2, y_2)$ for $x_1, x_2, y_1, y_2 \in G$. Now,

$$T_{f_{M \otimes P}(a,b)}[(x_1, y_1) \circ (x_2, y_2)] = T_{f_{M \otimes P}(a,b)}(x_1 \circ x_2, y_1 \circ y_2)$$

$$= T_{f_{M}(a)}(x_1 \circ x_2) * T_{f_{P}(b)}(y_1 \circ y_2)$$

$$\geq [T_{f_{M}(a)}(x_1) * T_{f_{M}(a)}(x_2)] * [T_{f_{P}(b)}(y_1) * T_{f_{P}(b)}(y_2)]$$

(as $M, P$ are two NSGs )

$$= [T_{f_{M}(a)}(x_1) * T_{f_{P}(b)}(y_1)] * [T_{f_{M}(a)}(x_2) * T_{f_{P}(b)}(y_2)]$$

( as $\ast$ is commutative )

$$= T_{f_{M \otimes P}(a,b)}(x_1, y_1) \ast T_{f_{M \otimes N}(a,b)}(x_2, y_2) \quad (10)$$

$$I_{f_{M \otimes P}(a,b)}[(x_1, y_1) \circ (x_2, y_2)] = I_{f_{M \otimes P}(a,b)}(x_1 \circ x_2, y_1 \circ y_2)$$

$$= I_{f_{M}(a)}(x_1 \circ x_2) \circ I_{f_{P}(b)}(y_1 \circ y_2)$$

$$\leq [I_{f_{M}(a)}(x_1) \circ I_{f_{M}(a)}(x_2)] \circ [I_{f_{P}(b)}(y_1) \circ I_{f_{P}(b)}(y_2)]$$

$$= [I_{f_{M}(a)}(x_1) \circ I_{f_{P}(b)}(y_1)] \circ [I_{f_{M}(a)}(x_2) \circ I_{f_{P}(b)}(y_2)]$$

$$= I_{f_{M \otimes N}(a,b)}(x_1, y_1) \circ I_{f_{M \otimes P}(a,b)}(x_2, y_2) \quad (11)$$

$$F_{f_{M \otimes P}(a,b)}[(x_1, y_1) \circ (x_2, y_2)] = F_{f_{M \otimes P}(a,b)}(x_1 \circ x_2, y_1 \circ y_2)$$

$$= F_{f_{M}(a)}(x_1 \circ x_2) \circ F_{f_{P}(b)}(y_1 \circ y_2)$$

$$\leq [F_{f_{M}(a)}(x_1) \circ F_{f_{M}(a)}(x_2)] \circ [F_{f_{P}(b)}(y_1) \circ F_{f_{P}(b)}(y_2)]$$

$$= [F_{f_{M}(a)}(x_1) \circ F_{f_{P}(b)}(y_1)] \circ [F_{f_{M}(a)}(x_2) \circ F_{f_{P}(b)}(y_2)]$$

$$= F_{f_{M \otimes P}(a,b)}(x_1, y_1) \circ F_{f_{M \otimes P}(a,b)}(x_2, y_2) \quad (12)$$

Hence the theorem follows from (10), (11) and (12).

It can be easily verified from Table 11 taking $u \ast v = uv$ and $u \circ v = u + v - uv$. 

□
Let identical. Every simple NSG is non-abelian unless all the neutrosophic soft elements of the group are identical. If all the neutrosophic soft elements of the NSG $\phi$ over $(G, E)$ is said to be a simple NSG if it has no neutrosophic normal soft subgroup other than $\phi_G$.

**Example 3.7.**
1. In the Example 3.1, the NSG $P$ is not simple.
2. The NSG $Q$ defined in Table 10 is a simple NSG as it has only a neutrosophic normal soft subgroup $\phi_{S_1}$.

**Theorem 3.6.** Every simple NSG is non-abelian unless all the neutrosophic soft elements of the group are identical.

**Proof.** Let $P$ be a simple NSG defined over $(G, E)$ whose at least two neutrosophic soft elements are non-identical. If all the neutrosophic soft elements of $P$ are identical, then $P$ is abelian whatever $G$ is (abelian / non-abelian). For contrary, suppose $P$ is abelian. Then $G$ is abelian by Remark 3.2. Let $M(\neq \phi_G)$ be a neutrosophic soft subgroup of $P$ over $(G, E)$. Then $M$ is an abelian NSG defined over $(G, E)$ and so is a normal neutrosophic soft subgroup of $P$ by Remark 3.3. Thus $P$ being a simple NSG has a normal neutrosophic soft subgroup $M(\neq \phi_G)$. This is a contradiction. Hence $P$ is non-abelian.

---

**Table 11.** Table for direct product $M \otimes P$

<table>
<thead>
<tr>
<th></th>
<th>$(f_M(a), f_P(a))$</th>
<th>$(f_M(a), f_P(b))$</th>
<th>$(f_M(a), f_P(c))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1)$</td>
<td>$(0.5, 0.6, 0.4)$</td>
<td>$(0.5, 0.6, 0.4)$</td>
<td>$(0.4, 0.6, 0.4)$</td>
</tr>
<tr>
<td>$(1, \omega)$</td>
<td>$(0.5, 0.6, 0.4)$</td>
<td>$(0.4, 0.6, 0.4)$</td>
<td>$(0.5, 0.6, 0.4)$</td>
</tr>
<tr>
<td>$(1, \omega^2)$</td>
<td>$(0.5, 0.7, 0.4)$</td>
<td>$(0.5, 0.6, 0.4)$</td>
<td>$(0.5, 0.6, 0.4)$</td>
</tr>
<tr>
<td>$(\omega, 1)$</td>
<td>$(0.4, 0.7, 0.4)$</td>
<td>$(0.4, 0.7, 0.4)$</td>
<td>$(0.4, 0.7, 0.4)$</td>
</tr>
<tr>
<td>$(\omega, \omega)$</td>
<td>$(0.4, 0.7, 0.4)$</td>
<td>$(0.4, 0.7, 0.4)$</td>
<td>$(0.4, 0.7, 0.4)$</td>
</tr>
<tr>
<td>$(\omega, \omega^2)$</td>
<td>$(0.4, 0.7, 0.4)$</td>
<td>$(0.4, 0.7, 0.4)$</td>
<td>$(0.4, 0.7, 0.4)$</td>
</tr>
<tr>
<td>$(\omega^2, 1)$</td>
<td>$(0.3, 0.8, 0.5)$</td>
<td>$(0.3, 0.8, 0.5)$</td>
<td>$(0.3, 0.8, 0.5)$</td>
</tr>
<tr>
<td>$(\omega^2, \omega)$</td>
<td>$(0.3, 0.8, 0.5)$</td>
<td>$(0.3, 0.8, 0.5)$</td>
<td>$(0.3, 0.8, 0.5)$</td>
</tr>
<tr>
<td>$(\omega^2, \omega^2)$</td>
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<td>$(0.3, 0.8, 0.5)$</td>
<td>$(0.3, 0.8, 0.5)$</td>
</tr>
</tbody>
</table>

---

**Table 12.** Table for direct product $M \otimes P$

<table>
<thead>
<tr>
<th></th>
<th>$(f_M(b), f_P(a))$</th>
<th>$(f_M(b), f_P(b))$</th>
<th>$(f_M(b), f_P(c))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1)$</td>
<td>$(0.3, 0.6, 0.7)$</td>
<td>$(0.3, 0.6, 0.7)$</td>
<td>$(0.3, 0.6, 0.7)$</td>
</tr>
<tr>
<td>$(1, \omega)$</td>
<td>$(0.3, 0.6, 0.7)$</td>
<td>$(0.3, 0.6, 0.7)$</td>
<td>$(0.3, 0.6, 0.7)$</td>
</tr>
<tr>
<td>$(1, \omega^2)$</td>
<td>$(0.3, 0.7, 0.7)$</td>
<td>$(0.3, 0.6, 0.7)$</td>
<td>$(0.3, 0.6, 0.7)$</td>
</tr>
<tr>
<td>$(\omega, 1)$</td>
<td>$(0.3, 0.8, 0.6)$</td>
<td>$(0.3, 0.8, 0.6)$</td>
<td>$(0.3, 0.8, 0.6)$</td>
</tr>
<tr>
<td>$(\omega, \omega)$</td>
<td>$(0.3, 0.8, 0.6)$</td>
<td>$(0.3, 0.8, 0.6)$</td>
<td>$(0.3, 0.8, 0.6)$</td>
</tr>
<tr>
<td>$(\omega, \omega^2)$</td>
<td>$(0.3, 0.8, 0.6)$</td>
<td>$(0.3, 0.8, 0.6)$</td>
<td>$(0.3, 0.8, 0.6)$</td>
</tr>
<tr>
<td>$(\omega^2, 1)$</td>
<td>$(0.4, 0.6, 0.7)$</td>
<td>$(0.4, 0.6, 0.7)$</td>
<td>$(0.4, 0.6, 0.7)$</td>
</tr>
<tr>
<td>$(\omega^2, \omega)$</td>
<td>$(0.4, 0.6, 0.7)$</td>
<td>$(0.4, 0.6, 0.7)$</td>
<td>$(0.4, 0.6, 0.7)$</td>
</tr>
<tr>
<td>$(\omega^2, \omega^2)$</td>
<td>$(0.4, 0.7, 0.7)$</td>
<td>$(0.4, 0.6, 0.7)$</td>
<td>$(0.4, 0.6, 0.7)$</td>
</tr>
</tbody>
</table>

---

**Table 13.** Table for direct product $M \otimes P$

<table>
<thead>
<tr>
<th></th>
<th>$(f_M(c), f_P(a))$</th>
<th>$(f_M(c), f_P(b))$</th>
<th>$(f_M(c), f_P(c))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1)$</td>
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<td>$(0.2, 0.4, 0.4)$</td>
<td>$(0.2, 0.3, 0.4)$</td>
</tr>
<tr>
<td>$(1, \omega)$</td>
<td>$(0.2, 0.5, 0.4)$</td>
<td>$(0.2, 0.5, 0.4)$</td>
<td>$(0.2, 0.3, 0.4)$</td>
</tr>
<tr>
<td>$(1, \omega^2)$</td>
<td>$(0.2, 0.7, 0.4)$</td>
<td>$(0.2, 0.6, 0.4)$</td>
<td>$(0.2, 0.3, 0.4)$</td>
</tr>
<tr>
<td>$(\omega, 1)$</td>
<td>$(0.5, 0.6, 0.6)$</td>
<td>$(0.5, 0.4, 0.6)$</td>
<td>$(0.4, 0.3, 0.6)$</td>
</tr>
<tr>
<td>$(\omega, \omega)$</td>
<td>$(0.5, 0.5, 0.6)$</td>
<td>$(0.4, 0.5, 0.6)$</td>
<td>$(0.5, 0.2, 0.6)$</td>
</tr>
<tr>
<td>$(\omega, \omega^2)$</td>
<td>$(0.5, 0.7, 0.6)$</td>
<td>$(0.5, 0.6, 0.6)$</td>
<td>$(0.5, 0.2, 0.6)$</td>
</tr>
<tr>
<td>$(\omega^2, 1)$</td>
<td>$(0.3, 0.6, 0.4)$</td>
<td>$(0.3, 0.4, 0.4)$</td>
<td>$(0.3, 0.3, 0.4)$</td>
</tr>
<tr>
<td>$(\omega^2, \omega)$</td>
<td>$(0.3, 0.5, 0.4)$</td>
<td>$(0.3, 0.5, 0.4)$</td>
<td>$(0.3, 0.3, 0.4)$</td>
</tr>
<tr>
<td>$(\omega^2, \omega^2)$</td>
<td>$(0.3, 0.7, 0.4)$</td>
<td>$(0.3, 0.6, 0.4)$</td>
<td>$(0.3, 0.3, 0.4)$</td>
</tr>
</tbody>
</table>
Theorem 3.7. Every non-abelian NSG is simple.

Proof. Let $M(\neq \phi_G)$ be a neutrosophic soft subgroup of a non-abelian NSG $P$ defined over $(G,E)$. Since $P$ is a non-abelian NSG, then $G$ is non-abelian classical group and so $M$ is non-abelian NSG over $(G,E)$. It implies $M$ is non-normal, otherwise a non-abelian NSG $P$ contains a normal neutrosophic soft subgroup $M(\neq \phi_G)$ which contradicts the Remark 3.3. Hence $P$ is simple as $M$ is arbitrary.

4. Conclusion

In the present paper, the concept of NSC, $N_{NSG}$, neutrosophic soft quotient group, direct product of NSGs and simple NSG have been proposed in a new approach. These are illustrated with suitable examples also. Several related properties and structural characteristics are investigated. Some theorems have been established and verified by suitable examples. We extend these concepts in Nss theory context and expect further work in this setting.

References
