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On Some Properties of the Sigmoid Function

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Abstract: By using some analytical techniques, we establish some properties of the sigmoid function. The properties are in the form of inequalities involving the function. Some of these inequalities connect the sigmoid function to the softplus function.

Key words: Sigmoid function, logistic function, softplus function, inequality, MN-convex function

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1. Introduction

The sigmoid function, which is also known as the standard logistic function is defined as

$$S(x) = \frac{e^x}{1+e^x} = \frac{1}{1+e^{-x}}, \quad x \in (-\infty, \infty), \tag{1}$$

$$= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right), \quad x \in (-\infty, \infty). \tag{2}$$

Its first and second derivatives are given as

$$S'(x) = \frac{e^x}{(1+e^x)^2} = S(x)\left(1 - S(x)\right), \tag{3}$$

$$S''(x) = \frac{e^x(1-e^x)}{(1+e^x)^3} = S(x)(1-S(x))(1-2S(x)), \tag{4}$$

for all $x \in (-\infty, \infty)$. It follows from (3) that $S(x)$ is increasing on $(-\infty, \infty)$. Also, in view of (3), $y = S(x)$ is a solution to the autonomous differential equation

$$\frac{dy}{dx} = y(1-y), \tag{5}$$

with initial condition $y(0) = 0.5$. Furthermore, the sigmoid function satisfies the following properties.

$$S(x) + S(-x) = 1, \tag{6}$$

$$S'(x) = S(x)S(-x), \tag{7}$$

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$$S'(x) = S'(-x), \tag{8}$$

$$\lim_{x\to\infty} S(x) = 1, \tag{9}$$

$$\lim_{x\to 0} S(x) = \frac{1}{2}, \tag{10}$$

$$\lim_{x\to -\infty} S(x) = 0, \tag{11}$$

$$\lim_{x\to \pm\infty} S'(x) = 0, \tag{12}$$

$$\lim_{x\to 0} S'(x) = \frac{1}{4}, \tag{13}$$

$$\int S(x)\,dx = \ln(1+e^x) + C, \tag{14}$$

where C is a constant of integration. The function $\ln(1+e^x)$ is known in the literature as softplus function [9]. It is clear from (14) that, the derivative of the softplus function gives the sigmoid function.

The sigmoid function has found useful applications in many scientific disciplines including machine learning, probability and statistics, biology, ecology, population dynamics, demography, and mathematical psychology (see [3] , [14], and the references therein).

In particular, the function is widely used in artificial neural networks, where it serves as an activation function at the output of each neuron (see [4], [5], [6], [10], [15]). Also, in the business field, the function has been applied to study performance growth in manufacturing and service management (see [13]). Another area of application is in the field of medicine, where the function is used to model the growth of tumors or to study pharmacokinetic reactions (see [14]). It is also applied in forestory. For example in [7], a generalized form of the function is applied to predict the site index of unmanaged loblolly and slash pine plantations in East Texas. Furthermore, it also applied in computer graphics or image processing to enhance image contrast (see [8], [12]).

The above important roles of the function makes its properties a matter of interest and hence worth studying. In the recent work [11], the authors studied some analytic properties of the function such as starlikeness and convexity in a unit disc.

In this paper, we continue the investigation. In the form of inequalities, we establish several properties of the sigmoid function. We begin with the following definitions and lemmas.

2. Auxiliary Definitions and Lemmas

Definition 2.1. A function $M : (0,\infty)\times(0,\infty) \to (0,\infty)$ is called a mean function if it satisfies the following conditions.

(i) $M(x,y) = M(y,x)$,

(ii) $M(x,x) = x$,

(iii) $x < M(x,y) < y$, for $x < y$,

(iv) $M(\lambda x, \lambda y) = \lambda M(x, y)$, for $\lambda > 0$.

There are several well-known mean functions in the literature. Amongst these are the following.

(i) Arithmetic mean: $A(x, y) = \frac{x+y}{2}$,

(ii) Geometric mean: $G(x, y) = \sqrt{xy}$,

(iii) Harmonic mean: $H(x, y) = \frac{1}{A(\frac{1}{x}, \frac{1}{y})} = \frac{2xy}{x+y}$,

(iv) Logarithmic mean: $L(x, y) = \frac{x-y}{\ln x - \ln y}$ for $x \neq y$, and $L(x, x) = x$.

(v) Identric mean: $I(x, y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}$ for $x \neq y$, and $I(x, x) = x$.

Definition 2.2 ([1])**.** Let $f : I \subseteq (0, \infty) \to (0, \infty)$ be a continuous function and M and N be any two mean functions. Then f is said to be MN-convex (MN-concave) if

$$f(M(x, y)) \leq (\geq) N(f(x), f(y)),$$

for all $x, y \in I$.

Lemma 2.1 ([1])**.** *Let $f : I \subseteq (0, \infty) \to (0, \infty)$ be a differentiable function. Then*

(a) f is GG-convex (or GG-concave) on I if and only if $\frac{xf'(x)}{f(x)}$ is increasing (or decreasing) for all $x \in I$.

(b) f is AH-convex (or AH-concave) on I if and only if $\frac{f'(x)}{f(x)^2}$ is increasing (or decreasing) for all $x \in I$.

(c) f is HH-convex (or HH-concave) on I if and only if $\frac{x^2 f'(x)}{f(x)^2}$ is increasing (or decreasing) for all $x \in I$.

Lemma 2.2 ([2])**.** *Let $f : (a, \infty) \to (-\infty, \infty)$ with $a \geq 0$. If the function defined by $g(x) = \frac{f(x)-1}{x}$ is increasing on (a, ∞), then the function $h(x) = f(x^2)$ satisfies the Grumbaum-type inequality*

$$1 + h(z) \geq h(x) + h(y), \tag{15}$$

where $x, y \geq a$ and $z^2 = x^2 + y^2$. If g is decreasing, then the inequality (15) is reversed.

Lemma 2.3 ([17])**.** *Let $-\infty \leq a < b \leq \infty$ and f and g be continuous functions that are differentiable on (a, b), with $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$. Suppose that $g(x)$ and $g'(x)$ are nonzero for all $x \in (a, b)$. If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing) on (a, b), then $\frac{f(x)}{g(x)}$ is also increasing (or decreasing) on (a, b).*

3. Main Results

Theorem 3.1. *The function $S(x)$ is subadditive on $(-\infty, \infty)$. In other words, the function satisfies the inequality*

$$S(x + y) < S(x) + S(y), \tag{16}$$

for all $x, y \in (-\infty, \infty)$.

First Proof. The case where $x = y = 0$ is trivial. Hence we only prove the result for the case where $x, y \in (0, \infty)$ and the case where $x, y \in (-\infty, 0)$. For $x \in (0, \infty)$, let $u(x) = \frac{e^x}{(1+e^x)^2}$. Then $u'(x) = \frac{e^x(1-e^x)}{(1+e^x)^3} < 0$ which implies that $u(x)$ is decreasing. Next let

$$\begin{aligned} f(x, y) &= S(x + y) - S(x) - S(y) \\ &= \frac{e^{x+y}}{1 + e^{x+y}} - \frac{e^x}{1 + e^x} - \frac{e^y}{1 + e^y}, \end{aligned}$$

for $x, y \in (0, \infty)$. Without loss of generality, let y be fixed. Then

$$\frac{\partial}{\partial x} f(x, y) = \frac{e^{x+y}}{(1 + e^{x+y})^2} - \frac{e^x}{(1 + e^x)^2} < 0,$$

since $u(x)$ is decreasing. Hence $f(x, y)$ is decreasing. Then for $x \in (0, \infty)$ we have

$$f(x, y) < f(0, y) = \lim_{x \to 0} f(x, y) = -\frac{1}{2} < 0,$$

which gives the desired result (16).

Likewise, for $x \in (-\infty, 0)$, let $w(x) = \frac{e^x}{(1+e^x)^2}$. Then $w'(x) = \frac{e^x(1-e^x)}{(1+e^x)^3} > 0$ which implies that $w(x)$ is increasing. Furthermore, let

$$g(x, y) = S(x + y) - S(x) - S(y),$$

for $x, y \in (-\infty, 0)$. Then for a fixed y we have

$$\frac{\partial}{\partial x} g(x, y) = \frac{e^{x+y}}{(1 + e^{x+y})^2} - \frac{e^x}{(1 + e^x)^2} > 0,$$

since $w(x)$ is increasing. Hence $g(x, y)$ is increasing. Then for $x \in (-\infty, 0)$ we have

$$g(x, y) < g(0, y) = \lim_{x \to 0} g(x, y) < 0.$$

Therefore, inequality (16) holds for all $x, y \in (-\infty, \infty)$. □

Second Proof. For $x > 0$, let $\phi(x) = \frac{x}{1+x}$. Then ϕ is increasing. Hence,

$$\begin{aligned} S(x) + S(y) = \frac{e^x}{1 + e^x} + \frac{e^y}{1 + e^y} &= \frac{e^x + e^y + 2e^{x+y}}{1 + e^x + e^y + e^{x+y}} \\ &> \frac{e^x + e^y + e^{x+y}}{1 + e^x + e^y + e^{x+y}} \\ &= \phi(e^x + e^y + e^{x+y}) \\ &> \phi(e^{x+y}) \\ &= S(x + y), \end{aligned}$$

which gives the desired result. □

Third Proof. By direct computation, we obtain

$$\left(\frac{S(x)}{x}\right)' = \frac{xe^x + xe^{2x} - xe^x - e^{2x} - xe^{2x}}{x^2(1+e^x)^2} = -\frac{e^{2x}}{x^2(1+e^x)^2} < 0,$$

for all $x \in (-\infty, \infty)$. Thus the function $\frac{S(x)}{x}$ is decreasing on $(-\infty, \infty)$. Then by virtue of Lemma 3.2 of [18], we conclude that $S(x)$ is subadditive on $(-\infty, \infty)$. □

Theorem 3.2. *The function $S(x)$ is logarithmically concave on $(-\infty, \infty)$. In other words, the inequality*

$$S\left(\frac{x}{a} + \frac{y}{b}\right) \geq [S(x)]^{\frac{1}{a}}[S(y)]^{\frac{1}{b}}, \tag{17}$$

is satisfied for $x, y \in (-\infty, \infty)$, where $a > 1$, $b > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$.

Proof. It suffices to show that $(\ln S(x))'' \leq 0$ for all $x \in (-\infty, \infty)$. Let $x \in (-\infty, \infty)$. Then we obtain

$$(\ln S(x))'' = \left(\frac{S'(x)}{S(x)}\right)' = -\frac{e^x}{(1+e^x)^2} \leq 0,$$

which concludes the proof. □

Corollary 3.1. *The inequalities*

$$S''(x)S(x) - (S'(x))^2 \leq 0, \quad x \in (-\infty, \infty), \tag{18}$$

$$S(1+u)S(1-u) \leq \left(\frac{e}{1+e}\right)^2, \quad u \in (-\infty, \infty), \tag{19}$$

are satisfied.

Proof. Inequality (18) is a direct consequence of the logarithmic concavity of $S(x)$. Then by letting $a = b = 2$, $x = 1+u$ and $y = 1-u$ in (17), we obtain (19). □

Theorem 3.3. *The function $S(x)$ satisfies the following inequalities.*

$$\frac{2e}{1+e} < \frac{S(x+1)}{S(x)} < e, \quad x \in (-\infty, 0), \tag{20}$$

$$1 < \frac{S(x+1)}{S(x)} < \frac{2e}{1+e}, \quad x \in (0, \infty). \tag{21}$$

Proof. Recall that $\left(\frac{S'(x)}{S(x)}\right)' \leq 0$, for all $x \in (-\infty, \infty)$. This means that the function $\frac{S'(x)}{S(x)}$ is decreasing for all $x \in (-\infty, \infty)$. Now, let $Q(x) = \frac{S(x+1)}{S(x)}$ for $x \in (-\infty, \infty)$. Then

$$Q'(x) = \frac{S(x+1)}{S(x)}\left\{\frac{S'(x+1)}{S(x+1)} - \frac{S'(x)}{S(x)}\right\} \leq 0,$$

since $\frac{S'(x)}{S(x)}$ is decreasing. This implies that $Q(x)$ is decreasing. Hence for $x \in (-\infty, 0)$, we obtain

$$\frac{2e}{1+e} = \lim_{x\to 0} Q(x) < Q(x) < \lim_{x\to -\infty} Q(x) = e,$$

which gives inequality (20). Also for $x \in (0, \infty)$, we obtain

$$1 = \lim_{x\to\infty} Q(x) < Q(x) < \lim_{x\to 0} Q(x) = \frac{2e}{1+e},$$

which gives inequality (21). This completes the proof. □

Theorem 3.4. *Let $\lambda = 1.2784645\ldots$ be the unique solution of the equation $1 + e^x - xe^x = 0$. Then the function $S(x)$ is GG-convex on $(0, \lambda)$ and GG-concave on (λ, ∞). That is,*

$$S(x^{\frac{1}{a}} y^{\frac{1}{b}}) \le [S(x)]^{\frac{1}{a}} [S(y)]^{\frac{1}{b}}, \tag{22}$$

for $x, y \in (0, \lambda)$ and

$$S(x^{\frac{1}{a}} y^{\frac{1}{b}}) \ge [S(x)]^{\frac{1}{a}} [S(y)]^{\frac{1}{b}}, \tag{23}$$

for $x, y \in (\lambda, \infty)$, where $a > 1$, $b > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$. Equality holds when $x = y$.

Proof. Let $f(x) = 1 + e^x - xe^x$ and $\lambda = 1.2784645\ldots$ be the unique solution of $1 + e^x - xe^x = 0$. Then $f(x) > 0$ if $x \in (0, \lambda)$ and $f(x) < 0$ if $x \in (\lambda, \infty)$. Also,

$$\frac{xS'(x)}{S(x)} = \frac{x}{1+e^x},$$

and so,

$$\left(\frac{xS'(x)}{S(x)}\right)' = \frac{1 + e^x - xe^x}{(1+e^x)^2}.$$

Hence $\frac{xS'(x)}{S(x)}$ is increasing on $(0, \lambda)$ and decreasing on (λ, ∞). Then by virtue of Lemma 2.1 (a), we conclude that $S(x)$ is GG-convex on $(0, \lambda)$ and GG-concave on (λ, ∞). These respectively imply inequalities (22) and (23). □

Theorem 3.5. *The function $S(x)$ is AH-concave on $(0, \infty)$. That is,*

$$S\left(\frac{x+y}{2}\right) \ge \frac{2S(x)S(y)}{S(x)+S(y)}, \tag{24}$$

for all $x, y \in (0, \infty)$. Equality holds when $x = y$.

Proof. We have

$$\left(\frac{S'(x)}{S(x)^2}\right)' = -\frac{1}{e^x} < 0,$$

for all $x \in (0, \infty)$, which implies that $\frac{S'(x)}{S(x)^2}$ is decreasing on $(0, \infty)$. Hence by Lemma 2.1 (b), we conclude that $S(x)$ is AH-convex on $(0, \infty)$. □

Theorem 3.6. *The function $S(x)$ is HH-convex on $(0,2)$ and HH-concave on $(2,\infty)$. That is,*

$$S\left(\frac{2xy}{x+y}\right) \leq \frac{2S(x)S(y)}{S(x)+S(y)}, \tag{25}$$

for $x,y \in (0,2)$ and

$$S\left(\frac{2xy}{x+y}\right) \geq \frac{2S(x)S(y)}{S(x)+S(y)}, \tag{26}$$

for $x,y \in (2,\infty)$. Equality holds when $x=y$.

Proof. We have

$$\frac{x^2 S'(x)}{S(x)^2} = \frac{x^2}{e^x},$$

which implies that

$$\left(\frac{x^2 S'(x)}{S(x)^2}\right)' = \frac{x(2-x)}{e^x}.$$

Then $\frac{x^2 S'(x)}{S(x)^2}$ is increasing if $x \in (0,2)$ and decreasing if $x \in (2,\infty)$. Hence by Lemma 2.1 (c), we conclude that $S(x)$ is HH-convex on $(0,2)$ and HH-concave on $(2,\infty)$. These respectively imply inequalities (25) and (26). $\square$

Theorem 3.7. *The function $S(x)$ satisfies the Grumbaum-type inequality*

$$1 + S(z^2) \geq S(x^2) + S(y^2), \tag{27}$$

where $x,y \in (0,\infty)$ and $z^2 = x^2 + y^2$.

Proof. Let $g(x)$ be defined for $x \in (0,\infty)$ as $g(x) = \frac{S(x)-1}{x}$. That is,

$$g(x) = \frac{\frac{e^x}{1+e^x} - 1}{x} = -\frac{1}{x(1+e^x)}.$$

Then

$$g'(x) = \frac{1}{x^2(1+e^x)} + \frac{e^x}{x(1+e^x)} > 0,$$

which implies that $g(x)$ is increasing. Hence by applying Lemma 2.2, we obtain the desired result (27). $\square$

In what follows we give some sharp inequalities connecting the sigmoid and the softplus functions.

Theorem 3.8. *The inequalities*

$$\frac{e^x}{1+e^x} < \ln(1+e^x) < \ln 2 - \frac{1}{2} + \frac{e^x}{1+e^x}, \quad x \in (-\infty,0), \tag{28}$$

$$\ln 2 - \frac{1}{2} + \frac{e^x}{1+e^x} < \ln(1+e^x), \quad x \in (0,\infty), \tag{29}$$

$$\frac{e^x}{1+e^x} < \ln(1+e^x), \quad x \in (-\infty, \infty), \tag{30}$$

are valid.

Proof. Let $F(x) = \ln(1+e^x) - \frac{e^x}{1+e^x}$ for all $x \in (-\infty, \infty)$. Then

$$F'(x) = \frac{e^x}{1+e^x}\left(1 - \frac{1}{1+e^x}\right) = \left(\frac{e^x}{1+e^x}\right)^2 > 0.$$

Thus $F(x)$ is increasing for all $x \in (-\infty, \infty)$. Then for $x \in (-\infty, 0)$, we have

$$0 = \lim_{x\to-\infty} F(x) < F(x) < \lim_{x\to 0} F(x) = \ln 2 - \frac{1}{2},$$

which gives inequality (28). For $x \in (0, \infty)$, we have

$$\ln 2 - \frac{1}{2} = \lim_{x\to 0} F(x) < F(x) < \lim_{x\to\infty} F(x) = \infty,$$

which gives inequality (29). Finally, for $x \in (-\infty, \infty)$, we have

$$0 = \lim_{x\to-\infty} F(x) < F(x) < \lim_{x\to\infty} F(x) = \infty,$$

which gives inequality (30). This completes the proof. □

Lemma 3.1. *The inequality*

$$e^x - \ln(1+e^x) > 0 \tag{31}$$

holds for all $x \in (-\infty, \infty)$.

Proof. Let $T(x) = e^x - \ln(1+e^x)$ for all $x \in (-\infty, \infty)$. Then

$$T'(x) = e^x\left(1 - \frac{1}{1+e^x}\right) = \frac{e^{2x}}{1+e^x} > 0,$$

which means that $T(x)$ is increasing. Then we have

$$\infty = \lim_{x\to\infty} T(x) > T(x) > \lim_{x\to-\infty} T(x) = \lim_{x\to-\infty} [e^x - \ln(1+e^x)] = 0,$$

which gives inequality (31). □

Theorem 3.9. *Let $f(x) = (1+e^x)^{\frac{1}{e^x}}$ and $g(x) = (1+e^x)^{1+\frac{1}{e^x}}$ for all $x \in (-\infty, \infty)$. Then $f(x)$ is decreasing and $g(x)$ is increasing. Consequently the inequalities*

$$(\ln 2)e^x < \ln(1+e^x) < e^x, \quad x \in (-\infty, 0), \tag{32}$$

$$\frac{e^x}{1+e^x} < \ln(1+e^x) < (2\ln 2)\frac{e^x}{1+e^x}, \quad x \in (-\infty, 0), \tag{33}$$

$$\frac{e^x}{1+e^x} < \ln(1+e^x) < e^x, \quad x \in (-\infty, \infty), \tag{34}$$

are satisfied.

Proof. Let $K(x) = \ln f(x) = \frac{\ln(1+e^x)}{e^x}$ and $L(x) = \ln g(x) = \frac{1+e^x}{e^x} \ln(1+e^x)$ for all $x \in (-\infty, \infty)$. Then

$$\begin{aligned} K'(x) &= \frac{1}{1+e^x} - \frac{\ln(1+e^x)}{e^x} \\ &= \frac{1}{e^x}\left[\frac{e^x}{1+e^x} - \ln(1+e^x)\right] < 0, \end{aligned}$$

which follows from (30). Hence $K(x)$ is decreasing and consequently, $f(x)$ is also decreasing. Also, we have

$$\begin{aligned} L'(x) &= 1 - \frac{\ln(1+e^x)}{e^x} \\ &= \frac{1}{e^x}[e^x - \ln(1+e^x)] > 0, \end{aligned}$$

which follows from Lemma 3.1. Thus $L(x)$ is increasing and consequently, $g(x)$ is also increasing. Moreover, we have

$$K(0) = \ln 2, \tag{35}$$

$$\lim_{x\to-\infty} K(x) = \lim_{x\to-\infty} \frac{\ln(1+e^x)}{e^x} = \lim_{x\to-\infty} \frac{1}{1+e^x} = 1, \tag{36}$$

$$\lim_{x\to\infty} K(x) = \lim_{x\to\infty} \frac{1}{1+e^x} = 0, \tag{37}$$

$$L(0) = 2\ln 2, \tag{38}$$

$$\lim_{x\to-\infty} L(x) = \lim_{x\to-\infty} \frac{\ln(1+e^x)}{\left(\frac{e^x}{1+e^x}\right)} = \lim_{x\to-\infty} (1+e^x) = 1, \tag{39}$$

$$\lim_{x\to\infty} L(x) = \lim_{x\to\infty} \frac{\ln(1+e^x)}{\left(\frac{e^x}{1+e^x}\right)} = \infty. \tag{40}$$

Since $K(x)$ is decreasing and $L(x)$ is increasing, we obtain the following. For $x \in (-\infty, 0)$, we have

$$\ln 2 = K(0) < K(x) < \lim_{x\to-\infty} K(x) = 1,$$

which gives inequality (32). Also, for $x \in (-\infty, 0)$, we have

$$1 = \lim_{x\to-\infty} L(x) < L(x) < L(0) = 2\ln 2,$$

which gives inequality (33). Furthermore, for $x \in (-\infty, \infty)$, we have

$$0 = \lim_{x\to\infty} K(x) < K(x) < \lim_{x\to-\infty} K(x) = 1,$$

which gives

$$\ln(1+e^x) < e^x. \tag{41}$$

Also, we have

$$1 = \lim_{x\to-\infty} L(x) < L(x) < \lim_{x\to\infty} L(x) = \infty,$$

which gives

$$\frac{e^x}{1+e^x} < \ln(1+e^x). \tag{42}$$

Now, by combining (41) and (42), we obtain (34). □

Theorem 3.10. *Let* Ψ *be defined for* $x \in (-\infty, 0)$ *by*

$$\Psi(x) = \frac{e^x \ln(1+e^x)}{e^x - \ln(1+e^x)}.$$

Then $\Psi(x)$ *is increasing and consequently, the inequality*

$$0 < \frac{e^x \ln(1+e^x)}{e^x - \ln(1+e^x)} < \frac{\ln 2}{1-\ln 2}, \tag{43}$$

is satisfied.

Proof. To begin with, we have

$$\lim_{x\to 0} \Psi(x) = \frac{\ln 2}{1-\ln 2},$$

and

$$\begin{aligned}
\lim_{x\to-\infty} \Psi(x) &= \lim_{x\to-\infty} \frac{e^x \ln(1+e^x)}{e^x - \ln(1+e^x)} \\
&= \lim_{x\to-\infty} \frac{\ln(1+e^x) - \frac{e^x}{1+e^x}}{\frac{e^x}{1+e^x}} \\
&= \lim_{x\to-\infty} \frac{\frac{e^x}{1+e^x} - \frac{e^x}{(1+e^x)^2}}{\frac{e^x}{(1+e^x)^2}} \\
&= \lim_{x\to-\infty} e^x \\
&= 0.
\end{aligned}$$

Next, let $f(x) = e^x \ln(1+e^x)$ and $g(x) = e^x - \ln(1+e^x)$. Then $f(-\infty) = \lim_{x\to-\infty} f(x) = 0$ and $g(-\infty) = \lim_{x\to-\infty} g(x) = 0$. Also,

$$f'(x) = e^x \left[\ln(1+e^x) + \frac{e^x}{1+e^x}\right],$$

and

$$g'(x) = e^x \left[1 - \frac{1}{1+e^x}\right] = e^x \frac{e^x}{1+e^x}.$$

Then

$$\frac{f'(x)}{g'(x)} = \frac{\ln(1+e^x) + \frac{e^x}{1+e^x}}{\frac{e^x}{1+e^x}} = \frac{\ln(1+e^x)}{\left(\frac{e^x}{1+e^x}\right)} - 1 = \frac{1+e^x}{e^x}\ln(1+e^x) - 1,$$

which implies that

$$\left(\frac{f'(x)}{g'(x)}\right)' = \left(\frac{1+e^x}{e^x}\ln(1+e^x)\right)' = \frac{1}{e^x}[e^x - \ln(1+e^x)] > 0.$$

Thus $\frac{f'(x)}{g'(x)}$ is increasing. Hence in view of Lemma 2.3, we conclude that $\frac{f(x)}{g(x)} = \Psi(x)$ is increasing. Then for $x \in (-\infty, 0)$ we have

$$0 = \lim_{x\to-\infty}\Psi(x) < \Psi(x) < \lim_{x\to 0}\Psi(x) = \frac{\ln 2}{1-\ln 2},$$

which yields inequality (43). □

Remark 3.1. *Let* $\alpha = \frac{\ln 2}{1-\ln 2}$. *Then inequality* (43) *can be rearranged as*

$$\ln(1+e^x) < \frac{\alpha e^x}{\alpha + e^x}, \tag{44}$$

for all $x \in (-\infty, 0)$.

4. Conclusion

In the form of inequalities, we have established several properties of the sigmoid function which is frequently used in artificial neural networks as well as some other scientific disciplines. The established results may find applications in the numerous areas where the sigmoid function is employed.

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