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One-point Ultra-F Compactification and *Stone*-$\check{C}$ech Ultra-F compactification of L-topological Spaces

Hanchuan LU[1,2*], Wenqing FU[3]
[1]School of Mathematical Sciences, Nanjing Normal University,
Nanjing, P.R. China
[2]School of Mathematics and Statistics, Guizhou University,
Guiyang, P.R. China
[3]School of Science, Xi'an Technological University,
Xi'an, P.R. China

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Abstract: In this paper, a method of getting Ultra-F compactification and *Stone*-$\check{C}$ech Ultra-F compactification of L-topology is given. We give the necessary and sufficient conditions of *Stone*-$\check{C}$ech Ultra-F compactification of L-topological spaces, and prove that it is unique in the sense of homeomorphism. Further, it is proved that an $L-$topology space is locally Ultra-F compact and Hausdorff if and only if its one-point compactificated space is Ultra-F compact and Hausdorff.

Key words: L-topology, *Stone*-$\check{C}$ech compactification, Ultra-F compactification, one-point Ultra-F Compactification

1. Introduction and Preliminaries

In 1965, in order to deal with the ambiguity of the uncertainty in mathematics, Zadeh [1] introduced the concept of fuzzy sets. Its emergence promotes the development of many subjects such as engineering mathematics, mathematics, computer and so on. Its application has been seen in many areas. For example, clustering analysis, automatic control, artificial intelligence, management decision-making, image recognition, system evaluation, and even the humanities and social sciences. In 1968, Chang [2] introduced the concept of fuzzy topological space by using fuzzy sets. Since then, Liu [3] and Wang [4] et. al. extended fuzzy topological space to L-topological space, where L is a complete lattice. Compactification is one of the important contents in topology, especially the study of the *Stone*-$\check{C}$ech compactification in topology is a basic work. Therefore, many scholars are interested in the research of *Stone*-$\check{C}$ech compactification [5–10]. We first give the definition of closure of an L-topological space, and then give the definition of F-compactification and ultra-F compactification and other related concepts by the definition of closure. At last, we mainly discuss the problem of one-point ultra-F compactification and *Stone*-$\check{C}$ech ultra-F compactification of L-topological Spaces. In the following, we give the preliminaries of this paper, readers can find the details in references [3, 4].

Let X be a set, if L is a Hutton Algebra (i.e. a completely distributive lattice with an order reversing involution$'$), L^X represents the set of all maps from X to L (which are called $L-$subsets of X). L^X forms a Hutton algebra under the pointwise order. For a set $Y \subseteq X$, and an $a \in L$, we define a mapping a_Y as $a_Y = a$ if $x \in Y$, or $a_Y = 0$ if $x \in X - Y$. Let $a, b \in L$, if for each $K \subseteq L$ satisfying the condition $\bigvee K \geq b$,

*Correspondence: luhanchuan2004@163.com

there is a $k \in K$ such that $k \geq a$, then we call a is triangle less than b, which is denoted by $a \lhd b$ or $b \rhd a$ (see[1]). For each $a \in L$, we denote $\Uparrow a = \{b \in L \mid b \rhd a\}$, $\Downarrow a = \{b \in L \mid b \lhd a\}$. It is easy to see when $L = [0,1]$, $a \lhd b$ if and only if $a < b$ $(\forall a, b \in [0,1])$. In addition, it can be proved that $a = \bigvee \Downarrow a$ $(\forall a \in L)$ if L is completely distributive lattice. For each $a \in L$ and each $\mu \in L^X$, $\mu_{\langle a \rangle}$ and $\mu_{[a]}$ are expressed as $\mu_{\langle a \rangle} = \{x \in X \mid \mu(x) \rhd a\}$, $\mu_{[a]} = \{x \in X \mid \mu(x) \ntriangleleft a\}$. And $\mu|_Y$ is used to express the limitations of μ in Y $(Y \subseteq X)$. An L-topology spaces is a pair $(X, \mathscr{T})$, where X is a set and $\mathscr{T} \subseteq L^X$, $\mathscr{T}$ is closed under arbitrary unions and finite inter-sections. Thus, we can obtain $0_X, 1_X \in \mathscr{T}$. When $L = [0,1]$, $(X, \mathscr{T})$ is also called fuzzy topological space. For an L-topology spaces $(X, \mathscr{T})$, $\hat{\imath}_L(\mathscr{T})$ represents the minimal topology on X containing the set $\{\mu_{\langle a \rangle} \mid \mu \in \mathscr{T}, a \in L\}$. When $L = [0,1]$, we tend to omit the subscript of $\hat{\imath}_L$. We call a member of $\mathscr{T}$ an L-open set. If L has an order reversing involution $' : L \longrightarrow L$, then we call μ' an L-closed set $(\forall \mu \in \mathscr{T})$. Among them, $\mu' \in L^X$, defined as $\mu'(x) = (\mu(x))'$. We write the closure of an L-subset $\mu \in L^X$ for μ^-. If $(X, \hat{\imath}_L(\mathscr{T}))$ is a compact space (resp., a local compact space, a Hausdorff space, a completely regular space), then $(X, \mathscr{T})$ is called an ultra-F compact L-topological space(resp., an ultra-F local compact L-topological space, an ultra-F Hausdorff L-topological space, an ultra-F completely regular L-topological space).

Let $f : X \longrightarrow Y$ be a mapping. Then an L-fuzzy mapping is induced by f as usual, i.e., $f_L^{\rightarrow}(A)(y) = \bigvee\{A(x) \mid f(x) = y\}$ $(\forall A \in L^X, \forall y \in Y)$. We call such a mapping f an L-forward powerset operator or L-valued Zadeh function. The right adjoint of $f_L^{\rightarrow}$ is represented as $f_L^{\leftarrow}$. It is also called L-backward power set operator. For any $B \in L^Y$, $f_L^{\leftarrow}(B) = \bigvee\{A \in L^X \mid f_L^{\rightarrow}(A) \leq B\} = B \circ f$. When $L = 2 = \{0,1\}$, the subscript of right adjoint of the $f_L^{\rightarrow}$ is often omitted. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be L-topological spaces, $f : X \longrightarrow Y$ is called continuous, if $f_L^{\leftarrow}(\nu) \in \mathscr{T}_X$ $(\forall \nu \in \mathscr{T}_Y)$. If $(Z, \mathscr{T}_Z)$ is an ultra-F compact L-topological space and $(X, \mathscr{T}_X)$ is a dense subspace of $(Z, \mathscr{T}_Z)$, then we call $(Z, \mathscr{T}_Z)$ an ultra-F compactification of $(X, \mathscr{T}_X)$. Similarly, If $(Z, \mathscr{T}_Z)$ is an ultra-F compact L-topological space, and ultra-F Hausdorff $L-$topology space $(X, \mathscr{T}_X)$ is a dense subspace of $(Z, \mathscr{T}_Z)$, then we call $(Z, \mathscr{T}_Z)$ an ultra-F compactification of ultra-F Hausdorff of $(X, \mathscr{T}_X)$. $(X, \mathscr{T}_X)$ called the dense subspace of the L-topological space $(Z, \mathscr{T}_Z)$ means that the closure of 1_X in $(Z, \mathscr{T}_Z)$ is equal to 1_Z. Let $(Y, \mathscr{T}_Y)$ be an ultra-F compactification of ultra-F Hausdorff $L-$topology space and f be a continuous mapping from $(X, \mathscr{T}_X)$ to $(Y, \mathscr{T}_Y)$, if there is a unique continuous mapping $\bar{f} : (Z, \mathscr{T}_Z) \longrightarrow (Y, \mathscr{T}_Y)$ makes $f = \bar{f}|_X$, then we call $(Z, \mathscr{T}_Z)$ a *Stone*-čech Ultra-F compactification of $(X, \mathscr{T}_X)$.

2. Main results

Lemma 2.1. *Let $f : X \longrightarrow Y$ be a mapping. Then*

(1) $f^{\leftarrow}(\mu_{\langle a \rangle}) = (f_L^{\leftarrow}(\mu))_{\langle a \rangle}$ and $f^{\leftarrow}(\mu_{[a]}) = (f_L^{\leftarrow}(\mu))_{[a]}$ $(\forall \mu \in L^Y, \forall a \in L)$.

(2) If $(X, \mathcal{J})$ is a topological space, $\mathcal{B} \subseteq 2^Y$ (2^Y is the power set of Y), and f satisfies the condition $f^{\leftarrow}(B) \in \mathcal{J}$ $(\forall B \in \mathcal{B})$, then $f : (X, \mathcal{J}) \longrightarrow (Y, \langle\langle \mathcal{B} \rangle\rangle)$ is a continuous map, where $\langle\langle \mathcal{B} \rangle\rangle$ is the smallest topology containing $\mathcal{B}$.

Proof. (1) In fact, $f^{\leftarrow}(\mu_{\langle a \rangle}) = f^{\leftarrow}[\mu^{\leftarrow}(\Uparrow a)] = (\mu \circ f)^{\leftarrow}(\Uparrow a) = (f_L^{\leftarrow}(\mu))^{\leftarrow}(\Uparrow a) = (f_L^{\leftarrow}(\mu))_{\langle a \rangle}$, $f^{\leftarrow}(\mu_{[a]}) = f^{\leftarrow}[\mu^{\leftarrow}(L- \Downarrow a)] = (\mu \circ f)^{\leftarrow}(L- \Downarrow a) = (f_L^{\leftarrow}(\mu))^{\leftarrow}(L- \Downarrow a) = (f_L^{\leftarrow}(\mu))_{[a]}$.

(2) It can be proved that $\mathcal{B} \cup \{X\}$ is a subbase of $\langle\langle \mathcal{B} \rangle\rangle$. Therefore, it is easy to get the conclusion. □

Lemma 2.2. *[8] Let X be a topological space, Z be a Hausdorff space, $A \subset X$. If $f : A \longrightarrow Z$ is continuous,* *then there is at most one continuous expansion* $g:\overline{A}\longrightarrow Z$ *of* f.

First, we give an ultra-F compactification of L-topological space and a kind of *Stone*-*č*ech ultra-F compactification.

Theorem 2.1. *Let* $(X,\mathscr{T})$ *be an* L*-fuzzy topological space.*

(1) If $(BX,\mathcal{J})$ *is a compactification of* $(X,\hat{\iota}_L(\mathscr{T}))$*, then* $(BX,\mathscr{T}_{\mathcal{J}})$ *is an ultra-F compactification of* $(X,\mathscr{T})$*. Here* $\mathscr{T}_{\mathcal{J}}=\{\mu\in L^{BX}\mid \mu|_X\in\mathscr{T},\mu_{\langle a\rangle}\in\mathcal{J}\ (\forall a\in L)\}$.

(2) Let $(X,\mathscr{T})$ *be both an ultra-F Hausdorff and ultra-F completely regular* L*-topological space,* $(BX,\mathcal{J})$ *be the Stone-čech compactification of* $(X,\hat{\iota}_L(\mathscr{T}))$*, then* $(BX,\mathscr{T}_{\mathcal{J}})$ *is a Stone-čech ultra-F compactification of* $(X,\mathscr{T})$*, that is* $(BX,\mathscr{T}_{\mathcal{J}})$ *is an ultra-F Hausdorff* L*-topological space and for any ultra-F Hausdorff and ultra-F compact* L*-fuzzy topological space* $(Y,\mathscr{T}_Y)$ *and any continuous mapping* $f:(X,\mathscr{T})\longrightarrow(Y,\mathscr{T}_Y)$*, there exists a unique continuous mapping* $\bar{f}:(BX,\mathscr{T}_{\mathcal{J}})\longrightarrow(Y,\mathscr{T}_Y)$ *that makes* $f=\bar{f}|_X$.

(3) An L*-topological space* $(X,\mathscr{T})$ *has a Stone-čech ultra-F compactification if and only if* $(X,\mathscr{T})$ *is ultra-F Hausdorff and ultra-F completely regular.*

(4) The Stone-čech Ultra-F compactification of $(X,\mathscr{T})$ *is unique in the sense of homeomorphism.*

Proof. (1) **Step 1** $(BX,\mathscr{T}_{\mathcal{J}})$ is an ultra-F compact L-topological space. Let $\{\nu_i\}_{i\in I}$ be a subset of $\mathscr{T}_{\mathcal{J}}$ and let $\nu=\bigvee_{i\in I}\nu_i$. For any $a\in L$, by $\nu|_X=\bigvee_{i\in I}(\nu_i|_X)$ and $\nu_{\langle a\rangle}=\bigcup_{i\in I}(\nu_i)_{\langle a\rangle}$ we can get $\nu\in\mathscr{T}_{\mathcal{J}}$. Similarly, $\mathscr{T}_{\mathcal{J}}$ is closed under finite intersections. So $(BX,\mathscr{T}_{\mathcal{J}})$ is an L-topological space. Next, from $\{\mu_{\langle a\rangle}\mid\mu\in\mathscr{T}_{\mathcal{J}},a\in L\}\subseteq\mathcal{J}$ we get $\hat{\iota}_L(\mathscr{T}_{\mathcal{J}})=\langle\langle\{\mu_{\langle a\rangle}\mid\mu\in\mathscr{T}_{\mathcal{J}},a\in L\}\rangle\rangle\subseteq\mathcal{J}$. Thus, $(BX,\hat{\iota}_L(\mathscr{T}_{\mathcal{J}}))$ is a compact space. That is, $(BX,\mathscr{T}_{\mathcal{J}})$ is an ultra-F compact L-topological space.

Step 2 $(X,\mathscr{T})$ is a subspace of $(BX,\mathscr{T}_{\mathcal{J}})$. For any $\omega\in\mathscr{T}_{\mathcal{J}}$, due to $\omega|_X\in\mathscr{T}$, it is only necessary to prove that when $\mu\in\mathscr{T}$, there exists a $\nu\in\mathscr{T}_{\mathcal{J}}$, such that $\mu=\nu|_X$.

First, for each $a\in L$, we make $V_a=\mu_{\langle a\rangle}$ and $V_a^*=BX-\overline{X-V_a}$, and define $\nu:BX\longrightarrow L$ as $\nu(x)=\bigvee\{a\in L\mid x\in V_a^*\}$. Here we refer to the closure of $X-V_a$ in $(BX,\mathcal{J})$ by $\overline{X-V_a}$. Then, for any $x\in X$ we can prove $\mu(x)=\bigvee\{a\in L\mid x\in V_a\}$.

Secondly, for each $a\in L$, we can prove $V_a=V_a^*\cap X$. Thus, we can get $\mu=\nu|_X$. Let $x\in V_a^*\cap X$. Then, there is an $x\in X$ and an open neighborhood G of x in $(BX,\mathcal{J})$ makes $G\cap(X-V_a)=\emptyset$. Therefore, $x\in V_a$. So, $V_a^*\cap X\subseteq V_a$. On the other hand,let $x\in V_a$, then $x\in X$. Hence, by the open set V_a^* in $(BX,\mathcal{J})$ and $V_a^*\cap(X-V_a)=V_a^*\cap X\cap(X-V_a)\subseteq V_a\cap(X-V_a)=\emptyset$, we get $x\in BX-cl_{BX}(X-V_a)=V_a^*$. so, $V_a\subseteq V_a^*\cap X$. Thus, for any $a\in L$, we prove that $V_a=V_a^*\cap X$.

Finally, for each $a\in L$, We show that $\nu_{\langle a\rangle}$ is an open set in $(BX,\mathcal{J})$. Thus by $\mu=\nu|_X$, can be $\nu\in\mathscr{T}_{\mathcal{J}}$. It is easy to see $x\in\nu_{\langle a\rangle}\iff\nu(x)=\bigvee\{b\in L\mid x\in V_b^*\}\rhd a\iff\exists\ d_x\in L$, s.t, $x\in V_{d_x}^*$ and $d_x\rhd a$. Let $x\in\nu_{\langle a\rangle}$. let $x\in\nu_{\langle a\rangle}$, then for each $y\in V_{d_x}^*$, by $\nu(y)=\bigvee\{b\in L\mid y\in V_b^*\}$, we can get $d_x\in\{b\in L\mid y\in V_b^*\}$. So as to know $d_x\le\nu(y)$.At this time, by $d_x\rhd a$ we can know $\nu(y)\rhd a$.This shows that $y\in\nu_{\langle a\rangle}$ is $x\in V_{d_x}^*\subseteq\nu_{\langle a\rangle}$. Because $V_{d_x}^*$ is an open subset of $(BX,\mathcal{J})$, so $\nu_{\langle a\rangle}$ is an open subset of $(BX,\mathcal{J})$.

Step 3 $(X,\mathscr{T})$ is a dense subspace of $(BX,\mathscr{T}_{\mathcal{J}})$. Let $\lambda\in\mathscr{T}_{\mathcal{J}}$ and $\lambda'\ge 1_X$, then for any $a\in L$, we can see that $BX-\lambda_{\langle a\rangle}=(\lambda')_{[a]}$ is a closed set of $(BX,\hat{\iota}_L(\mathscr{T}_{\mathcal{J}}))$ containing X. Because $(X,\hat{\iota}_L(\mathscr{T}))$ is a dense subspace of $(BX,\mathcal{J})$ and $\hat{\iota}_L(\mathscr{T}_{\mathcal{J}})\subseteq\mathcal{J}$, so for any $a\in L$, $(\lambda')_{[a]}=BX$. That is $\lambda_{\langle a\rangle}=\emptyset\ \ (\forall a\in L)$. So for any $x \in BX$, we have $\lambda'(x) = \bigwedge\{b \in L \mid b \rhd \lambda'(x)\} = \bigwedge\{b \in L \mid b' \lhd \lambda(x)\} = \bigwedge\limits_{x \in \lambda_{\langle b' \rangle}} b = \bigwedge \emptyset = 1$. That is $\lambda' = 1_{BX}$. This shows that $(X, \mathscr{T})$ is a dense subspace of $(BX, \mathscr{T}_{\mathcal{J}})$.

(2) **Step 1** $(BX, \hat{\iota}_L(\mathscr{T}_{\mathcal{J}}))$ is a Hausdorff space.

Here we only need to prove $\mathcal{J} \subseteq \hat{\iota}_L(\mathscr{T}_{\mathcal{J}})$. For any $U \in \mathcal{J}$, we define $\mu : BX \longrightarrow L$ as

$$\mu(x) = \begin{cases} 0, & x \in BX - U, \\ 1, & x \in U. \end{cases}$$

For any $a \in L$, $\mu_{\langle a \rangle} = \{x \in BX \mid \mu(x) \rhd a\} = U \in \mathcal{J}$. We give the definition of $\mu|_X : X \longrightarrow L$ as

$$(\mu|_X)(x) = \begin{cases} 0, & x \in X - U, \\ 1, & x \in X \cap U. \end{cases}$$

So, $(\mu|_X)_{\langle a \rangle} = X \cap U$. Also because $(X, \hat{\iota}_L(\mathscr{T}))$ is a subspace of $(BX, \mathcal{J})$, so $(\mu|_X)_{\langle a \rangle} \in \hat{\iota}_L(\mathscr{T})$. Thus $\mu|_X \in \mathscr{T}$. And then get $\mu \in \mathscr{T}_{\mathcal{J}}$. Hence $U = \mu_{\langle a \rangle} \in \hat{\iota}_L(\mathscr{T}_{\mathcal{J}})$. So we prove that $\mathcal{J} \subseteq \hat{\iota}_L(\mathscr{T}_{\mathcal{J}})$. For any $x, y \in BX$, $x \neq y$, because $(BX, \mathcal{J})$ is a Hausdorff space, so there exist $U, V \in \mathcal{J}$ such that $x \in U, y \in V$, and $U \cap V = \emptyset$. Since $\mathcal{J} \subseteq \hat{\iota}_L(\mathscr{T}_{\mathcal{J}})$, so $U, V \in \hat{\iota}_L(\mathscr{T}_{\mathcal{J}})$. Thus $(BX, \hat{\iota}_L(\mathscr{T}_{\mathcal{J}}))$ is a Hausdorff space.

Step 2 Let $(X, \hat{\iota}_L(\mathscr{T}))$ be a completely regular Hausdorff space, $(BX, \mathcal{J})$ be a *Stone*-*č*ech compactification of $(X, \hat{\iota}_L(\mathscr{T}))$ and $f : (X, \mathscr{T}) \longrightarrow (Y, \mathscr{T}_Y)$ be a continuous mapping, where $(Y, \mathscr{T}_Y)$ is an ultra-F Hausdorff and ultra-F compact L-fuzzy topological space. By lemma 2.1, we can see that $f : (X, \hat{\iota}_L(\mathscr{T})) \to (Y, \hat{\iota}_L(\mathscr{T}_Y))$ is a continuous map. So there is only one continuous mapping $\bar{f} : (BX, \mathcal{J}) \longrightarrow (Y, \hat{\iota}_L(\mathscr{T}_Y))$ makes $f = \bar{f}|_X$. Let $\mu \in \mathscr{T}_Y$. In the following, will prove $\bar{f}_L^{\leftarrow}(\mu) \in \mathscr{T}_{\mathcal{J}}$, which implies $(Y, \mathscr{T}_Y)$ is a continuous mapping. First, because of the continuity of $f : (X, \mathscr{T}) \longrightarrow (Y, \mathscr{T}_Y)$, $\bar{f}_L^{\leftarrow}(\mu)|_X = (\mu \circ \bar{f})|_X = \mu \circ f = f_L^{\leftarrow}(\mu) \in \mathscr{T}$. Secondly, since $\bar{f} : (BX, \mathcal{J}) \longrightarrow (Y, \hat{\iota}_L(\mathscr{T}_Y))$ is a continuous mapping, it can be known that $\left(\bar{f}_L^{\leftarrow}(\mu)\right)_{\langle a \rangle} = \bar{f}_L^{\leftarrow}(\mu_{\langle a \rangle}) \in \mathcal{J}$ $(\forall a \in L)$. Therefore, $\bar{f}_L^{\leftarrow}(\mu) \in \mathscr{T}_{\mathcal{J}}$.

Step 3 Let $g : (BX, \mathscr{T}_{\mathcal{J}}) \longrightarrow (Y, \mathscr{T}_Y)$ be a continuous map and meet the conditions of $f = g|_X$. Then by lemma 2.1, the $g : (BX, \hat{\iota}_L(\mathscr{T}_{\mathcal{J}})) \longrightarrow (Y, \hat{\iota}_L(\mathscr{T}_Y))$ is a continuous map. By $\hat{\iota}_L(\mathscr{T}_{\mathcal{J}}) \subseteq \mathcal{J}$ we know that $g : (BX, \mathcal{J}) \longrightarrow (Y, \hat{\iota}_L(\mathscr{T}_Y))$ is also a continuous mapping. Therefore, $g = \bar{f}$.

(3) $(X, \mathscr{T})$ is both an ultra-F Hausdorff space and a $T_{3.5}$ space $\Longleftrightarrow (X, \hat{\iota}_L(\mathscr{T}))$ is both a Hausdorff space and completely regular space. Topological space X has compactification if and only if X is completely regular $\Longleftrightarrow (X, \hat{\iota}_L(\mathscr{T}))$ has a *Stone*-*č*ech compactification $\Longleftrightarrow (X, \mathcal{J})$ has an ultra-F compactification.

(4) Let $(Y_1, \mathcal{J}_1)$ and $(Y_2, \mathcal{J}_2)$ be two *Stone*-*č*ech Ultra-F compactifications of $(X, \mathscr{T})$. We define $f_1 : X \longrightarrow Y_1$ as the identity map. By $(Y_1, \mathcal{J}_1)$ is a *Stone*-*č*ech ultra-F compactification of $(X, \mathscr{T})$, we can know that f_1 is a continuous map. Similarly, we give the definition of $f_2 : X \longrightarrow Y_2$ as the identity map. By $(Y_2, \mathcal{J}_2)$ is a *Stone*-*č*ech ultra-F compactification of $(X, \mathscr{T})$, we can know that f_2 is a continuous map too. Because of $(Y_1, \mathcal{J}_1)$ and $(Y_2, \mathcal{J}_2)$ are two *Stone*-*č*ech Ultra-F compactification of $(X, \mathscr{T})$, there is a continuous map $\overline{f_1} : Y_2 \longrightarrow Y_1$ and $\overline{f_2} : Y_1 \longrightarrow Y_2$, which satisfying $\overline{f_1}|_X = f_1$ and $\overline{f_2}|_X = f_2$. Then, $\overline{f_1} \circ \overline{f_2} : (Y_1, \mathcal{J}_1) \longrightarrow (Y_1, \mathcal{J}_1)$ is continuous, and $\overline{f_1} \circ \overline{f_2}|_X = f_1$. At the same time, the $id_{Y_1} : Y_1 \longrightarrow Y_1$ is continuous and $id_{Y_1} : Y_1|_X = f_1$. By lemma 2.1(1), it is easy to get $\overline{f_1} \circ \overline{f_2} : (Y_1, \hat{\iota}_L(\mathcal{J}_1)) \longrightarrow (Y_1, \hat{\iota}_L(\mathcal{J}_1))$ is continuous, and $id_{Y_1} : (Y_1, \hat{\iota}_L(\mathcal{J}_1)) \longrightarrow (Y_1, \hat{\iota}_L(\mathcal{J}_1))$ is also continuous. Since $(Y_1, \hat{\iota}_L(\mathcal{J}_1))$ is a Hausdorff space,

we can get $\overline{f_1} \circ \overline{f_2} = id_{Y_1}$ by lemma 2.2. Similarly, we can get $\overline{f_2} \circ \overline{f_1} = id_{Y_2}$. Therefore, we have $(Y_1, \mathcal{J}_1)$ and $(Y_2, \mathcal{J}_2)$ are homeomorphic. □

Following the notation of Theorem 2.1, when $(BX, \mathcal{J})$ is a one-point compactification of $(X, \hat{\iota}_L(\mathscr{T}))$, we call $(BX, \mathscr{T}_{\mathcal{J}})$ is the one-point ultra-F compactification of $(X, \mathscr{T})$. By general topology, we can get the one-point ultra-F compactification of $(X, \mathscr{T})$ always exists. By lemma 2.1, that one-point ultra-F compactification of $(X, \mathscr{T})$ is unique in the sense of homeomorphism. Next, we will give further results.

Theorem 2.2. *Let $(BX, \mathscr{T}_{\mathcal{J}})$ be a one-point ultra-F Compactification of $(X, \mathscr{T})$ and make $BX - X = \{\infty\}$.*

(1) $1_X \in \mathscr{T}_{\mathcal{J}}$ (That is, 1_X is an open L- subset of $(BX, \mathscr{T}_{\mathcal{J}})$). When $(X, \mathscr{T})$ is an ultra-F compact $L-$topological space, $1_{BX-X} \in \mathscr{T}_{\mathcal{J}}$ is also established (That is, 1_{BX-X} is an open and closed L-subset of $(BX, \mathscr{T}_{\mathcal{J}})$).

(2) $(BX, \mathscr{T}_{\mathcal{J}})$ is an ultra-F Hausdorff L-topological space $\Longleftrightarrow$ $(X, \mathscr{T})$ is an ultra-F locally compact ultra-F Hausdorff topological space.

Proof. (1) Because $(BX, \mathcal{J})$ is a one-point compactification of $(X, \hat{\iota}_L(\mathscr{T}))$, so $X \in \mathcal{J}$. According to the definition of $\mathscr{T}_{\mathcal{J}}$, we can know that $1_X \in \mathscr{T}_{\mathcal{J}}$. When $(X, \mathscr{T})$ is an ultra-F compact L-topological space, $(X, \hat{\iota}_L(\mathscr{T}))$ is a compact space. By $(BX, \mathscr{T}_{\mathcal{J}})$ is a one-point ultra-F compactification of $(X, \mathscr{T})$, then $\{\infty\} \in \mathcal{J}$. And according to the definition of $\mathscr{T}_{\mathcal{J}}$, so we can get $1_{\{\infty\}} \in \mathscr{T}_{\mathcal{J}}$.

(2) It is only need to prove $(BX, \hat{\iota}_L(\mathscr{T}_{\mathcal{J}}))$ is a Hausdorff space $\Longleftrightarrow$ $(X, \hat{\iota}_L(\mathscr{T}))$ is a locally compact Hausdorff space. By general topology knowledge and $\hat{\iota}_L(\mathscr{T}_{\mathcal{J}}) \subseteq \mathcal{J}$, the necessary condition is clearly established. The following prove the sufficient conditions is established. Let $(X, \hat{\iota}_L(\mathscr{T}))$ be a locally compact Hausdorff space, $x, y \in BX$ $(x \neq y)$. Thus we need to find an open set U containing x and an open set V containing y in $(BX, \hat{\iota}_L(\mathscr{T}_{\mathcal{J}}))$ with $U \cap V = \varnothing$. The following two cases are discussed.

Case 1 $x, y \in X$. Since $(X, \hat{\iota}_L(\mathscr{T}))$ is a Hausdorff space and lemma 2.1(2), there are two finite subsets $\mathscr{A}$ and $\mathscr{B}$ which has the form $\{\mu_{\langle a \rangle} \mid \mu \in \mathscr{T}, a \in L\}$. Without loss of generality, we let $\mathscr{A} = \{(\mu_1)_{\langle a \rangle}, (\mu_2)_{\langle b \rangle}\}$, $\mathscr{B} = \{(\nu_1)_{\langle c \rangle}, (\nu_2)_{\langle d \rangle}\}$. Thus there exist an open set $(\mu_1)_{\langle a \rangle} \cap (\mu_2)_{\langle b \rangle}$ containing x and an open set $(\nu_1)_{\langle c \rangle} \cap (\nu_1)_{\langle d \rangle}$ containing y with $(\mu_1)_{\langle a \rangle} \cap (\mu_2)_{\langle b \rangle} \cap (\nu_1)_{\langle c \rangle} \cap (\nu_1)_{\langle d \rangle} = \emptyset$. Take $\hat{\mu}_i, \hat{\nu}_i \in L^{BX}$, make $\hat{\mu}_i(\infty) = \hat{\nu}_i(\infty) = 0$, $\hat{\mu}_i|_X = \mu_i$, $\hat{\nu}_i|_X = \nu_i$ $(i = 1, 2)$. Since $(\hat{\mu}_1)_{\langle a \rangle} = (\mu_1)_{\langle a \rangle}$, $(\hat{\mu}_2)_{\langle b \rangle} = (\mu_2)_{\langle b \rangle}$, $(\hat{\nu}_1)_{\langle c \rangle} = (\nu_1)_{\langle c \rangle}$, $(\hat{\nu}_2)_{\langle d \rangle} = (\nu_2)_{\langle d \rangle}$. Therefore, $(\hat{\mu}_1)_{\langle a \rangle}, (\hat{\mu}_2)_{\langle b \rangle}, (\hat{\nu}_1)_{\langle c \rangle}, (\hat{\nu}_2)_{\langle d \rangle} \in \hat{\iota}_L(\mathscr{T}_{\mathcal{J}})$. Obviously, $U = (\hat{\mu}_1)_{\langle a \rangle} \cap (\hat{\mu}_2)_{\langle b \rangle}$ and $V = (\hat{\nu}_1)_{\langle c \rangle} \cap (\hat{\nu}_2)_{\langle d \rangle}$ meet the requirements.

Case 2 $x \in X, y = \infty$. Since $(X, \hat{\iota}_L(\mathscr{T}))$ is a locally compact space, there exist a compact neighborhood W in $(X, \hat{\iota}_L(\mathscr{T}))$ with $x \in W \subset X$. Because $(BX, \mathcal{J})$ is a one-point compactification of $(X, \hat{\iota}_L(\mathscr{T}))$, so $\{\infty\} \in \mathcal{J}$. By the definition of $\mathscr{T}_{\mathcal{J}}$, then $1_{\{\infty\}} \in \mathscr{T}_{\mathcal{J}}$. This shows that $\{\infty\} \in \hat{\iota}_L(\mathscr{T}_{\mathcal{J}})$. Since W is a compact neighborhood in $(X, \hat{\iota}_L(\mathscr{T}))$ with $x \in W \subset X$. So there is a finite subset $\mathscr{A}$ of $\{\mu_{\langle a \rangle} \mid \mu \in \mathscr{T}, a \in L\}$ with $x \in (\mu_1)_{\langle a \rangle} \cap (\mu_2)_{\langle b \rangle} \subseteq W$. Here, we assume that $\mathscr{A} = \{(\mu_1)_{\langle a \rangle}, (\mu_2)_{\langle b \rangle}\}$. Put $\hat{\mu}_i \in L^{BX}$ with $\hat{\mu}_i(\infty) = 0$, $\hat{\mu}_i|_X = \mu_i$ $(i = 1, 2)$. Since $(\hat{\mu}_1)_{\langle a \rangle} = (\mu_1)_{\langle a \rangle}$ and $(\hat{\mu}_2)_{\langle b \rangle} = (\mu_2)_{\langle b \rangle}$, so $(\hat{\mu}_1)_{\langle a \rangle}, (\hat{\mu}_2)_{\langle b \rangle} \in \hat{\iota}_L(\mathscr{T}_{\mathcal{J}})$. Let $U = (\hat{\mu}_1)_{\langle a \rangle} \cap (\hat{\mu}_2)_{\langle b \rangle}$. Then there an open neighborhood U of x and an open neighborhood $V\,(V = \{\infty\})$ of y in $(BX, \hat{\iota}_L(\mathscr{T}_{\mathcal{J}}))$ with $U \cap V = \emptyset$. □

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