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Computation of First-Fit Coloring of Graphs

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Abstract: The first-fit chromatic number of a graph is the maximum possible number of colors used in a first fit coloring of the graph. In this paper, we compute the first-fit chromatic number for a special class of bipartite graphs. Further we give a crisp description on the computational aspects of the first fit chromatic number and indicate the scope for further applications. We also raise some open problems.

Key words: Graph, coloring, chromatic number, First-Fit Chromatic Number

1. Introduction

A coloring or a proper coloring of a graph G is an assignment of positive integers called "colors" to the vertices of G so that adjacent vertices have different colors. The minimum number of colors used in a proper coloring is called the chromatic number of G, denoted by $\chi(G)$. A graph coloring can be a solution to some specific kind of problem. For instance, an animal preserving firm is destined to conserve a set V of animals with different compatibility levels among themselves. We distinguish an incompatible pair with an edge. A vertex coloring of the graph $G(V, E)$ allots enclosures to them. Coloring of such a graph with many colors are easy to obtain but we require a coloring with least number of colors. The name coloring hails from the 19th century challenge that whether four colors are sufficient to color a planar map of a country. Each state is a vertex, and every pair of states that share a common boundary becomes an edge, so a proper coloring gives them different colors. The identities of the colors are not pertinent mathematically, so what is desired is not a coloring, and is actually a graph partition that honors graph structure. Graph coloring is one of the most vital concepts in graph theory. It is employed in several practical applications of computer science such as clustering, data mining, image capturing, image segmentation, networking, resource allocation, processes scheduling etc.,

Several upper bounds on the chromatic number comes from algorithms that generates a coloring. The most basic algorithm is the greedy algorithm. A greedy coloring relative to a vertex ordering $\sigma = v_1 < v_2 < \cdots < v_n$ of $V(G)$ is obtained by coloring the vertices in the order $v_1, v_2, \ldots, v_n$ assigning to v_i the smallest positive integer not already used on its neighbours which are lower-indexed. The first-fit chromatic number, denoted by $\chi_{FF}(G)$, is the largest number of colors used such that G has a greedy coloring.

The study of first-fit chromatic number continues to attract the attention of researchers since a first-fit coloring problem occurs frequently in real life applications such as dynamic storage allocation problem [4, 7, 9, 10, 12, 16] and radio frequency allocation problem [19]. Historically, the first-fit chromatic number is also called the Grundy number. The study of Grundy coloring dates back to the 1930's when Grundy used them in the study of kernels of directed Graphs [11]. After that, many researchers have studied the first-fit chromatic number under different names [2, 6]. It is believed that Christen and Selkow were the first to define

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and study the first-fit chromatic number as a graph parameter [3]. For an interval graph I_k with maximum clique size k, it was shown that $\chi_{FF}(I_k) = O(k \log k)$ [18]. The logarithmic term of $O(k \log k)$ was eliminated by showing that $\chi_{FF}(I_k) \leq 40k$ [14]. After a series of papers [1, 13, 15, 17], it was shown that $\chi_{FF}(I_k) \leq 8k$. For the lower bound of the first-fit chromatic number of an interval graph I_k with maximum clique size k, it was shown that for every $\epsilon > 0$, $\chi_{FF}(I_k) > (5-\epsilon)k$, where k is sufficiently large.

The first-fit coloring can determine a proper coloring of a graph, but it is believed that it is difficult to find an efficient procedure to determine the first-fit chromatic number of a graph. Very recently, it was shown that the first-fit chromatic number problem of determining it for a given graph to be at least k, can be done in time $O^*(2p)$ for graphs of order p [5]. Although it was known that $\chi(G) \leq \chi_{FF}(G) \leq \Delta(G)+1$, the inequality $\chi(G) \leq \chi_{FF}(G)$ may be tight, and also be very loose. It was shown that for any fixed $k \geq 0$, given a graph G, it is co-NP complete to decide whether $\chi_{FF}(G) \leq \chi(G) + k$; given a graph G which is the complement of a bipartite graph, it is co-NP Complete to decide whether $\chi(G) = \chi_{FF}(G)$ [21]. In [8] the authors suggested a new method, stimulated by zero-knowledge proof systems, for establishing lower bounds on the chromatic number of a graph. To elaborate on this method they derived simple reductions from max-3-coloring and max-3-sat, establishing that it is hard to approximate the chromatic number within $\Omega(N^{\delta})$ for some $\delta > 0$.

In this paper, we compute the first-fit chromatic number of a special class of bipartite graph defined as below.

2. First Fit Coloring of a Special Graph

A special bipartite graph $G_{m,n}$ shown in Fig. 1 was given in [20], where $V(G_{m,n}) = \{u\} \cup \{u_1, \ldots, u_m\} \cup \{v\} \cup \{v_1, \ldots, v_n\} \cup \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$ and $E(G) = \{(u_i, x_j) | 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{(v_i, y_j) | 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{(x_i, y_j) | 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{(u_i, v_j) | 1 \leq i \leq j \leq n\} \cup \{(u, v_m), (v, y_n)\}$. We prove that the first-fit chromatic number of $G_{m,n}$ is 3.

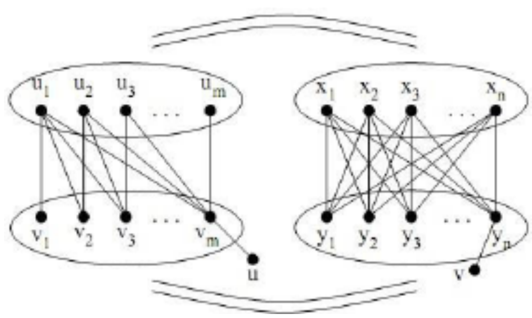

Figure 1. The bipartite graph $G_{m,n}$, where the parallel lines indicate the join operation

Yegnanarayanan [12] has constructed the above graph and proved the existence of a solution for the following question in [11]. For any three integers, a, b, c with $2 \leq a \leq b \leq c$, does there exists a graph G with $\chi(G) = a$, $\alpha(G) = b$ and $\psi(G) = c$. Motivated by this result we raise the following open problem.

Problem: For positive integer a, b, c, d with $2 \leq a \leq b \leq c \leq d$, does there exists a graph G such that $\chi(G) = a$, $\chi_{FF}(G) = b$, $\alpha(G) = c$ and $\psi(G) = d$. Here α and ψ denote respectively achromatic number and pseudoachromatic number. While attempting this problem we somehow ended with the following theorem.

Theorem 2.1. $\chi_{FF}(G_{m,n}) = 3$.

Proof.
The bipartite graph $G_{m,n}$ has the bipartition (V_1, V_2), where $V_1 = \{u\} \cup \{u_i | 1 \leq i \leq m\} \cup \{y_j | 1 \leq j \leq n\}$ and $V_2 = \{v\} \cup \{v_i | 1 \leq i \leq m\} \cup \{x_j | 1 \leq j \leq n\}$. For convenience, let $U = \{u_i | 1 \leq i \leq m\}$, $V = \{v_j | 1 \leq j \leq n\}$, $X = \{x_j | 1 \leq j \leq n\}$ and $Y = \{y_j | 1 \leq j \leq n\}$. Let f be a first-fit coloring of $G_{m,n}$.

Fact 1. For any first fit coloring f of $G_{m,n}$, the set V cannot hold at the same time a vertex with color 2 and a vertex with color higher than 2. Symmetrically, the same is true for the set U.

Proof of Fact 1.

We prove this fact by using reductio-ad-absurdam. Assume that $v_k, v_i \in V$ and $f(v_k) = 2$, $f(v_i) = c^*$ with $c^* > 2$. By the definition of first-fit coloring, the vertex v_i must be adjacent with a vertex having color 2. By the structure of $G_{m,n}$, this vertex with colore 2 cannot be in Y, so, there is a vertex $u_j \in U$ such that $f(u_j) = 2$ and $j \leq i$. It is easy to observe that $k < j$, since, otherwise, the coloring is not proper. Again by the definition of first-fit coloring, v_k must be adjacent with a vertex having color 1. Hence, there must be a vertex u_l such that $f(u_l) = 1$ and $l \leq k$. It is clear that the set Y cannot contain a vertex with color 1 (if Y contains a vertex with color 1, then the sets X and V cannot contain vertices with color 1, which means that the vertices of U are not adjacent with any vertex of color 1. However, it cannot be the case, as u_j has to be adjacent with a vertex having color 1). By the structure of $G_{m,n}$, this vertex with color 1 cannot also be in X. So, there should be some $v_h \in V$ such that $f(v_h) = 1$ and $(u_l, v_h) \in E(G_{m,n})$ with $h \geq j$. Therefore, $l \leq k \leq j \leq h$, which implies $l \leq h$ and contradicts the definition of first-fit coloring.

Fact 2. There is no first-fit coloring assigning 4 colors to the vertices in $U \cup V \cup \{u\}$.

Proof of Fact-2.

Assume that there is a first-fit coloring f assigning 4 colors to the vertices in $U \cup V \cup \{u\}$. As u is a pendant vertex and f is a first-fit coloring, it is not possible for f to assign a color more than 2 to u. This means $f(u) = 1$ or $f(u) = 2$. Suppose that $f(u) = 2$. Then $f(v_m) = 1$. See Fig. 2.

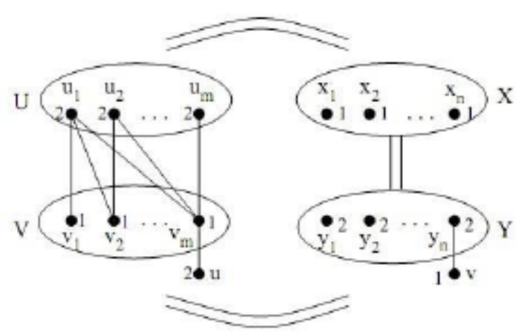

Figure 2.

As all u_i's $1 \leq i \leq m$ and all y_j's $1 \leq j \leq n$ are adjacent to v_m it is obvious that none of u_i's and none of y_j's are assigned the color 1 by f. But then the construction of $G_{m,n}$ reveals the fact that every v_i for $1 \leq i \leq m-1$ and every x_j for $1 \leq j \leq n$ are assigned the color 1 by f. This further implies that f has no choice but to assign the color 1 to v. So if $f(u) = 2$ then it is not possible for f to assign 4 colors to the vertices of $U \cup V \cup \{u\}$.

Next suppose that $f(u) = 1$ and the vertex v_m is not joined to any other vertex with color 1. This means neither u_i for $1 \leq i \leq m$ nor y_j for $1 \leq j \leq n$ can be colored with color 1. See Fig. 3.

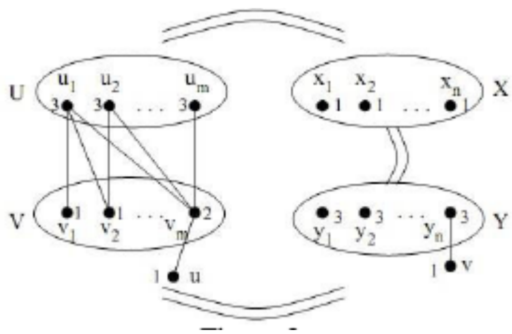

Figure 3.

So every x_j for $1 \leq j \leq n$ and all v_i, for $1 \leq i \leq m-1$ must be colored with color 1. As every u_i for $1 \leq i \leq m$ is adjacent only to v_m we deduce that all the u_i for $1 \leq i \leq m$ must be colored with either the color 2 or with the color 3. If $f(u_i) = 3$ for all $1 \leq i \leq m$ then $f(v_m) = 2$ and certainly $f(y_j) = 3$ for all $1 \leq j \leq n$. Now as v is a pendant vertex it cannot be colored with the color 4 by f as it is a first-fit coloring. This means $f(v) = 1$ or 2 and hence $U \leq V \leq \{u\}$ cannot get 4 colors under a first-fit coloring f with $f(u_i) = 3$.

If $f(u_i) = 2$ for all $1 \leq i \leq m$ then obviously $f(v_m) \neq 2$ and this forces all y_j for $1 \leq j \leq n$ to be colored with color 2 by f and with color 1 to v. See Fig. 4. Now as v_m is not adjacent to any vertex with color 3, it is clear that $f(v_m) \neq 4$ and hence $f(v_m) = 3$. Moreover it is not possible that $f(x_j) = 4$ for $1 \leq j \leq n$ by the construction of $G_{m,n}$. So $f(x_j) = 1$ or 3. Hence in this instance also $U \cup V \cup \{u\}$ cannot receive 4 colors under a first-fit coloring f.

To sum up we state that in view of Fact-1 and the argument put under Fact-2, it is not possible for a first-fit coloring f to assign 4 colors to the elements of $U \cup V \cup \{u\}$.

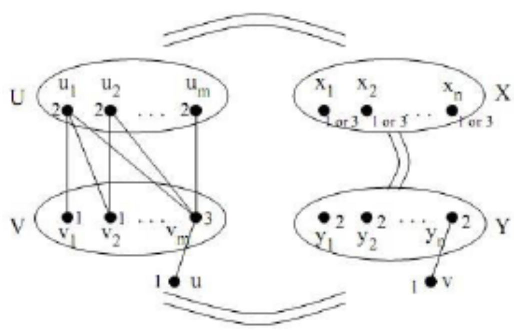

Figure 4.

Proof of Main Theorem.

Let f be a proper first-fit coloring of $G_{m,n}$ defined as follows: $f(u_1) = 2$, $f(v_1) = 1$; $f(u_i) = 1$ for $2 \leq i \leq m$; $f(v_i) = 3$ for $2 \leq i \leq m$; $f(x_j) = 3$ for $1 \leq j \leq n$; $f(y_j) = 2$ for $1 \leq j \leq n$; $f(u) = 1 = f(v)$. It is easy to see that $\chi_{FF}(G_{m,n}) \leq 3$. We next claim that $\chi_{FF}(G_{m,n}) \leq 3$. Suppose that $\chi_{FF}(G_{m,n}) = 4$. We derive a contradiction as follows:

We begin by noting from Fact-2 that the vertices of $U \cup V \cup \{u\}$ cannot contain among themselves four distinct colors. Moreover as v is also a pendant vertex like the vertex u, it also cannot be assigned the color 4 by a proper first-fit coloring f. So the only possibility is either set X has a vertex of color 4 or the set Y

has a vertex of color 4. Without loss of generality assume that Y includes a vertex of color 4 and in such an instance we derive a contradiction as follows. The other possibility that X includes a vertex of color 4 can be disposed off in a similar fashion.

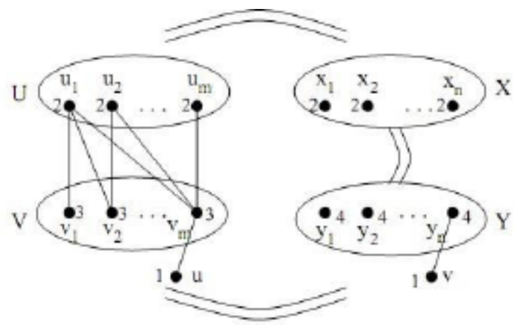

Figure 5.

By Fact-1 we infer that V cannot contain at a time vertices with color 2 and vertices with color 3. If V includes a vertex with color 3 then by Fact-1 u_i for $1 \leq i \leq m$ should be colored with color 2 and in that event due to the presence of all possible edges between U and X some vertex x_j in X should have the color 2 to be adjacent with a vertex of color 4 in Y. Now f can assign the color 1 to u and v. See Fig. 5. Even in the event of this occurrence a vertex with color 2 cannot be adjacent with a vertex of color 1 and hence f cannot be a first-fit coloring. This is a contradiction and so V cannot include within it a vertex with color 3.

If V includes a vertex with color 2 then two possibilities can occur. The set U containing a vertex of color 3 as one such possibility and the other is U not containing the color 3. In the event of the occurrence of the former the pattern of distribution of colors among U, V, X, Y, u and v are shown in Fig. 6.

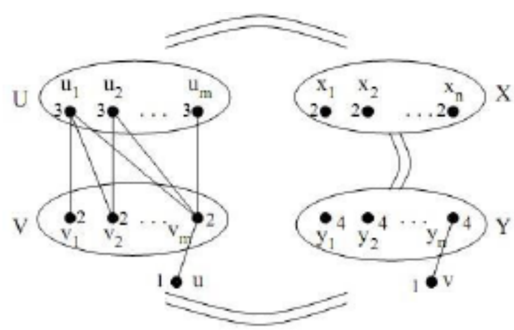

Figure 6.

As a vertex with color 4 is not adjacent with any vertex colored 3, f cannot be a first-fit coloring. This is a contradiction to our assumption that f is a first-fit coloring. So the possibility of U holding a vertex of color 3 cannot happen. In the event of occurrence of the latter, it is clear that the vertices of U can either contain a vertex with color 1 or a vertex with color 2. If U contains a vertex with color 1 then the resulting color distribution possibility is shown in Fig. 6. Now as a vertex with color 3 is not adjacent with any vertex with color 2, f cannot be a first-fit coloring. This is a contradiction to our assumption that f is a first-fit coloring. Suppose U contains a vertex with color 2 then that vertex in U cannot be adjacent with a vertex with color 2 in V. This means the vertex v_m cannot be colored with color 2. By Fact-1 the vertex v_m then cannot be colored with the colour higher than 2. So it has to be assigned with the color 1 only. Futher in this instant the vertex u cannot be colored with the color 1 and hence it has to be colored with color 2. The resulting color distribution possibility is shown in Fig. 7. Now we are in a fix as we cannot assign any color greater than 2 to any of u_i, $1 \leq i \leq m-1$ by Fact-2 and also the color 1 as f is a proper coloring. So clearly the event of U containing a vertex of color 2 cannot happen.

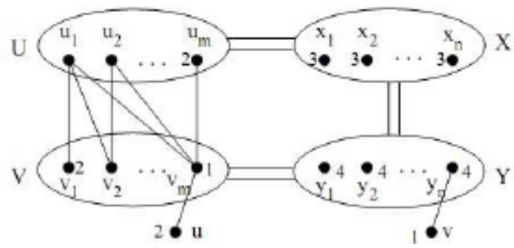

Figure 7.

Hence we conclude that $\chi(G_{m,n}) \leq 3$ and $\chi(G_{m,n}) = 3$. □

3. Application

Wireless communication is employed in a variety of real life problems such as mobile telephony, radio, TV broadcasting, satellite communication to name a few. In all these problems a frequency assignment problem is hidden with particular characteristics. Different modelling approaches are available in literature for each of the features of the problem, such as handling of interference among radio signals, the availability of frequencies, and the optimization criterion. In frequency assignment model to cellular antenna one can use conflict free coloring (CF), wherein a small number of unique frequencies in total are applied to allot to a large number of antennas in a wireless network. A cellular network is a heterogeneous network with two different types of nodes: base-stations and clients. The base-stations are linked among them by an external fixed backbone network. Clients are linked only to base stations; links between clients and base-stations are achieved by radio links. Fixed frequencies are allotted to base-stations to enable links to clients. Clients scan frequencies nonstop in search of a base-station with good reception features. This process takes place effortlessly and enables transitions in a smooth way between base-stations when a client is mobile. Look at a client who is within the reception range of two base stations. If each of these two base stations are allotted the same frequency, then it results in mutual interference and the links between the client and each of these conflicting base stations become noisy a lot to be used. Under normal cases a base station may serve a client provided the reception is strong and interference is weak when it I from other base stations ten it is weak The basic problem of frequency assignment in cellular network is to allot frequencies to base stations in such a manner that every client is served by some base station. The objective is to minimize the number of allotted frequencies as the available spectrum is limited and costly. This problem of frequency assignment was treated as a graph coloring problem, where the vertices of the graph are the given set of antennas and the edges are those pairs of antennas that overlap in their reception range. So if we color the vertices of the graph such that no two vertices that are connected by an edge with the same color then we can ensure that there are no conflicting base stations. But this model is very rare, if a client lies within the reception range of say, k antennas, then each pair of these antennas are conflicting and hence must be assigned k different colors. If one of these antennas is allotted a color 1 even if all other antennas are assigned the same color, 2 then we have a employ a total of two colors and this client can still be served.

Let $H = (V, E)$ be a hypergraph. A conflict-free coloring of H is an assignment of colors to V such that, in each hyperedge $e \in E$, there is at least one uniquely-colored vertex. This notion is an extension of the classical graph coloring. Such colorings arise in the context of frequency assignment to cellular antennae, in battery consumption aspects of sensor networks, in RFID protocols, and several other fields. Conflict-free coloring has been the focus of many recent research papers.

A natural question thus arises: Suppose we are given a set of n antennas. The location of each antenna (base station) and its radius of transmission is fixed and is known (and is modeled as a disc in the plane). We seek the least number of colors that always suffice such that each of the discs is assigned one of the colors and such that every covered point p is also covered by some disc D whose assigned color is distinct from all the colors of the other discs that cover p. This is a special case of CF-coloring where the underlying hypergraph is induced by a finite family of discs in the plane.

Open Problem

Compute the first fit chromatic number of the above said hypergraph.

4. Conclusion

We have successfully computed the first fit chromatic number of a special class of bipartite graph. We also discussed the various computationally relevant upper and lower bounds of first-fit chromatic number in terms of usual chromatic number and discussed its applications that are of practical interest. We also raise open problems which would open the gate for a flood of research outputs in the near future.

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