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On Coloring Distance Graphs

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Abstract: Some problems become very popular and demand rapt attention while others just collect dust on the shelf. Big Conferences like this are a nice breeding ground for advertising or publicizing both old and new problems. Consider for example the famous unit-distance problem. This problem asks for the smallest number of colors needed to color the points of the plane R^2 so that points unit distance apart have distinct colors. This problem was attributed to atleast five different mathematicians in a variety of combinations. Edward Nelson, Hugo Hadwiger, Paul Erdos, Martin Gardner and Leo Moser. It is now known as Hadwiger-Nelson problem (HNP). The problem was actually formulated by Edward Nelson in 1950 when he was a graduate student at the University of Chicago. Nelson called it the alternative four-color problem as it dealt with the plane and four colors. He proved that atleast four colors are needed. John Isbell upon learning about the problem from Nelson, proved in 1951 that the unit distance graph is 7-colorable. This problem might be a typical candidate for a famous problem. It is simple enough for high school student to understand, it is easy to describe and looks deceptively easy to solve. After all, all one needs is to construct a finite set of points in the plane that requires more than 4-colors or prove that any set of points is 4-colorable. As a matter of fact, while hundreds of papers dedicated to variations of this problem have been published, no progress has been made on the actual chromatic number of the unit distance graph since its inception. In this paper this problem is discussed further. Its variations, enormity, challenges, bottleneck and road ahead are quite mind-boggling. However, upon reading this one gets clarity regarding the role of several pure mathematics concepts emanating from set theory, algebra, analysis, topology and number theory. The author of this paper was fascinated very much and is completely engrossed with this problem since 2010. The following are some of his results in this topic.

Given a subset D in Z, an integer distance graph is a graph $G(Z, D)$ with Z as vertex set and with an edge joining two vertices u and v iff $|u - v| \in D$. The author considered the problem of determining $\chi(G(Z, D))$ when D is either a) a set of $(n + 1)$ positive integers for which the n^{th} power of the last is the sum of the n^{th} powers of the previous terms or b) a set of Pythagorean quadruples or c) a set of Pythagorean n-tuples or d) a set of square distances or e) a set of abundant numbers or deficient numbers or Carmichael numbers or f) a set of polytope numbers or g) a set of happy numbers or lucky numbers or h) a set of Lucas numbers or i) a set of Ulam numbers or j) a set of weird numbers. In addition, he also found some useful upper and lower bounds for general cases too technical to mention here [14].

Key words: Graphs, chromatic number, plane-coloring problem.

1. Introduction

A typical problem in Euclidean Ramsey theory is to determine the largest number p such that each coloring of the n-dim euclidean space with p colors contains a monochromatic congruent copy of a fixed finite configuration

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of points. The simplest configuration about which this question can be asked consists of two points unit distance apart.

The chromatic number of a space S is the minimum number of colors needed to color all the points of S so that no two points of the same color are at unit distance. A unit distance graph in space S is a graph that can be embedded in S so that the distance between each two adjacent vertices is 1.

The chromatic number $\chi(S)$ of the space S is equal to chromatic number of the graph whose vertices are all the points of the space and two vertices are joined by an edge iff the distance of their corresponding points is 1. This graph is an infinite unit ditance graph.

For any space S, $r \in R^+$, let $\chi(S,r)$ denote the chromatic number of the graph whose vertices are all the points in S and edges connect pair of vertices distance r apart. The distance r is then called the forbidden distance $\chi(S,1) = \chi(S)$.

The structure of the n-dim real space does not change after scaling. Therefore, instead of forbidding two monochromatic points to be distance 1 apart, we can equivalently use any $r \in R^+$ as the forbidden distance.

$$\forall\ r \in R^+ : \chi(R^n, r) = \chi(R^n)$$

The situation in the line is simple: the chromatic number is 2.

Theorem 1.1. $\chi(G(R, D = \{1\})) = 2$.

Proof. Partition the vertex set of G, into two non empty disjoint sets such that their union is R. That is, $V(R) = V_1 \cup V_2$, where $V_1 = \bigcup_{n=-\infty}^{\infty} [2n, 2n+1)$ and $V_2 = \bigcup_{n=-\infty}^{\infty} [2n+1, 2n+2)$. Now color all the vertices of V_1 with color 1 and the vertices of V_2 with color 2. As V_i, $i = 1, 2$ are independent and $G(R, \{1\})$ is bipartite the result follows. □

However the problem of finding the chromatic number of the plane is open. It is called Hadwiger-Nelson Problem (HNP) and was originated around 1950 by Edward Nelson [7].

Lemma 1.1 (Rado Lemma)**.** [11] *Let M and M_1 be arbitrary sets. Assume that to any $v \in M_1$ there corresponds a finite subset A_v of M. Assume that to any finite subset $N \subset M$, a choice function $x_N(v)$ is given, which attaches an element of A_v to each element v of N: $x_N(v) \in A_v$. Then there exists a choice function $x(v)$ defined for all $v \in M_1$ ($x(v) \in A_v$ if $v \in M_1$) with the following property. If K is any finite subset of M_1, then there exists a finite subset N ($K \subset N \subset M_1$), such that, as far as K is concerned, the function $x(v)$ coincides with $x_N(v) : x(v) = x_N(v)$ ($v \in K$).*

Clearly $G(R^2, \{1\})$ is an infinite graph. The problem of finding the chromatic number of $G(R^2, \{1\})$ is enormously difficult. Paul Erdos has mentioned this problem as one of his favourite problems. Although he could not solve this problem he has made a significant progress towards the solution of this problem with a vital result using Rado's lemma and is given as follows:

Theorem 1.2. [3] *Let k be a positive integer, and let the graph G have the property that any finite subgraph is k-colorable. Then G is k-colorable itself.*

The Hadwiger-Nelson Problem (HNP): What is the least number $\chi(R^2)$ of colors required to color the two-dimensional Euclidean plane R^2 so that points $x, y \in R^2$ which are unit distance apart gets distinct color.

Erdos-de Bruijn theorem says that if an infinite graph has a finite chromatic number, then it has a finite subgraph with the same chromatic number. Because the chromatic number of the n-dim real space is finite, its chromatic number is the maximum of the chromatic numbers of the finite unit distance graphs in the n-dim space.

Because of this result people started searching for finite unit distance subgraphs which require the maximum number of colors. Also see [1, 3, 6, 7, 12].

2. How to Generalize HNP to other Fields?

It is possible to generalize HNP by attempting to assign the colors to the points whose coordinates lie on a certain subfield E of R. That is, ask for $\chi(E^2)$ of the graph $G(E^2)$ whose vertices are the points of E^2 with an edge connecting any two points (x_1, x_2) and (y_1, y_2) whenever $(x_1 - y_1)^2 + (x_2 - y_2)^2 = 1$.

Let K be any field or any commutative ring with unity. Let $K^2 = K \times K$ be a graph in which two vertices $(a, b), (c, e) \in K^2$ are adjacent if and only if $(a - c)^2 + (b - d)^2 = 1$. It is naturally difficult to compute the $\chi(K^2)$. Led by an intuition that $\chi(R^2)$ may be equal to 7, researchers started looking for finite subgraphs $G \subset R^2$ with as few vertices as possible, and but without any proper 4-coloring. They soon realised that such subgraphs must have hundreds, if not thousands, of vertices. So it should be prudent to choose G so that available computational sources can be used to show that $\chi(G) > 4$.

Interestingly the task of finding $\chi(R^2)$ is related to finding $\chi(K^2)$ for other rings K as well. In particular, the values of $\chi(F^2)$, for finite fields F, play a significant role. Note that $2 \le \chi(K^2) \le \chi(L^2) \le 7$ for all subfields $K \subseteq L \subseteq R$. By Erdos-de Bruijn result, $\chi(R^2)$ is the maximum of $\chi(G)$ among all finite induced subgraphs $G \subset R^2$. As every such subgraph G has coordinates in a subfield $K \subsetneq R$ which is finitely generated over Q, we infer that $\chi(R^2)$ is the maximum of $\chi(K^2)$ among all finitely generated subfields $K \subseteq R$. This maximum is realised in fact for some number field K.

Two important configurations in the plane which are compass-constructible from Q^2 are the equilateral triangle of side length 1 and the Moser spindle graph.

Moser spindle graph named after Leo Moser and his brother William is an undirected simple graph with 7 vertices and 11 edges. It is also called Hajos graph as it can be viewed as an instance of Hajos construction. It can be constructed graph theoretically, without reference to geometric embedding, using Hajos construction starting with two complete graphs on 4 vertices. This construction removes an edge from each complete graph, merges two of the end points of the removed edges into a single vertex shared by both cliques, and adds a new edge connecting the remaining two end points of the removed edge. Another way of constructing is as the complement graph of the utility graph $K_{3,3}$ by subdividing one of its edges. See Figure 1.

Proposition 2.1. *The graph K^2 contains a 3-cycle if and only if K contains $\frac{1}{2}$, (That is, K does not have characteristic 2) and $\sqrt{3}$.*

Proof. If K contains $\frac{1}{2}$ and $\sqrt{3}$ then K^2 contains an equilateral triangle with vertices (0, 0), (1, 0) and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Conversely, suppose K^2 contains an equilateral triangle with vertices v_1, v_2 and $v_3 \in K^2$. Assume that $v_1 = (0, 0)$; otherwise one can translate $K^2 \to K^2$, be defining $v_1 \to v - v_1$. Next one can deem $v_2 = (1, 0)$; else, take $v_2 = (a, b)$ with $a, b \in K$. As $a^2 + b^2 = 1$, we can rotate $K^2 \to K^2$, $v \to v \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. [That is, if

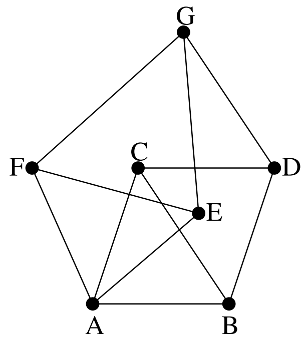

Figure 1.

$v_2 = (a,b)$ then $v_2 \to v_2 \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ means $(a,b)\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = (a^2+b^2, 0) = (1,0)]$. Lastly, take $v_3 = (c,d)$. As $c^2+d^2 = 1$ and $(c-1)^2+d^2 = 1$, we get $c^2-2c+1+d^2-c^2-d^2 = 0$. That is, $c = \frac{1}{2}$ and hence $d = \frac{\sqrt{3}}{2}$. This completes the proof. □

Proposition 2.2. *The graph K^2 contains a Moser spindle if and only if K contains $\frac{1}{66}$, $\sqrt{3}$ and $\sqrt{11}$.*

Proof. Look at the Moser spindle graph shown in Figure 1. If K contains $\frac{1}{66}$, $\sqrt{3}$ and $\sqrt{11}$, then let $A = (0,0)$, $B = (1,0)$ nd $C = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. If $D = (a,b)$ then we have $(a-1)^2+b^2 = 1$, $(a-\frac{1}{2})^2+(b-\frac{\sqrt{3}}{2})^2 = 1$. So $a^2-a+\frac{1}{4}+b^2-b\sqrt{3}+\frac{3}{4}-a^2+2a-1-b^2 = 0$. That is, $a = b\sqrt{3}$. This means $(b\sqrt{3}-\frac{1}{2})^2+(b-\frac{\sqrt{3}}{2})^2 = 1$. That is, $3b^2-b\sqrt{3}+\frac{1}{4}+b^2-b\sqrt{3}+\frac{3}{4} = 1$, $4b^2-2b\sqrt{3} = 0$. That is, $2b^2 = b\sqrt{3}$ or $b = \frac{\sqrt{3}}{2}$. This implies that $a^2-2a+1+\frac{3}{4} = 1$. That is, $a^2-2a+\frac{3}{4} = 0$. That is $a = \frac{2\pm\sqrt{4-3}}{2} = \frac{2\pm 1}{2} = \frac{3}{2}$ or $\frac{1}{2}$. Take $a = \frac{3}{2}$ and disregard $\frac{1}{2}$, we get $D = (\frac{3}{2}, \frac{\sqrt{3}}{2})$. A little more rigorous calculation reveals that $E = (\frac{5}{6}, \frac{\sqrt{11}}{6})$, $F = (\frac{5-\sqrt{33}}{12}, \frac{5\sqrt{3}+\sqrt{11}}{12})$ and $G = (\frac{15-\sqrt{33}}{12}, \frac{5\sqrt{3}+3\sqrt{11}}{12})$. The converse can be done as in the above proposition.

In view of the Proposition 2.1 we deduce that the smallest number field K for which K^2 contains a 3-cycle, is when, $K = Q(\sqrt{3})$. In this case, it is clear that $\chi(K^2) \geq 3$. In order to see the upper bound, we have to consider a real quadratic extension of Q, i.e., a filed of the term $K = Q(\sqrt{d})$ for some square free integer $d \geq 2$. We know that, for a prime $p \equiv 3 \pmod 4$ if $d \equiv 0 \pmod p$ or d is a quadratic residue modulo p, then $\chi(K^2) \leq \chi(F_p^2)$. Now let q be a prime power. Form a graph F_q^2 with adjacency relation as usual following euclidean ℓ_2-norm. Then the graph F_q^2 has q^2 vertices, and is regular of degree q, if q is even; $q-1$ if $q \equiv 1 \pmod 4$; and $q+1$ if $q \equiv 3 \pmod 4$. With some computer assistance one can show that $\chi(F_3^2) = 3$. So we get $\chi(K^2) \leq 3$. Hence $\chi(K^2) = 3$. □

3. Some Notes on the Determination of $\chi(G(R^2, \{1\}))$

It is obvious that $G(R^2, \{1\})$ is not a trivial graph. Therefore $\chi(G(R^2, \{1\}) \neq 1$. The presence of at least one edge, viz., between $(0,0)$ and $(1,1)$ in $G(R^2, \{1\})$ conveys the information that $\chi(G(R^2, \{1\})) \geq 2$. Similarly, the presence of a clique of size 3, viz., K_3 with vertices at $(0,0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(1,0)$ shows that $\chi(G(R^2, \{1\})) \geq 3$.

Moreover, it is a fact that four points in the euclidean two dimensional plane cannot have pairwise odd integer distances. Note that four points in a plane with pairwise odd integral distances do not exist. This is because of the following result by L. Piepmeyer.

Theorem 3.1. [10] *The maximum number $f(n)$ of odd integral distances between n points in the plane is*

$$f(n) = \frac{n^2}{3} + \frac{r(r-3)}{6},\ r = 1,2,3 \text{ and } n \equiv r \ (mod\ 3).$$

Therefore, a clique of size 4, viz., K_4 cannot be an induced subgraph of $G(R^2, \{1\})$. But it will be quite interesting to find the coordinates of a unit distance finite subgraph G_1 of $G(R^2, \{1\})$ such that $\chi(G_1) = 4$.

The Moser spindle graph is a smallest such graph with chromatic number 4. Then it is interesting to note that so far no unit distance graph that requires exactly five colors are known. Falconer [6] showed that if the color classes form measurable sets of R^2, then atleast five colors are needed. Since the construction of non-measurable sets requires the axiom of choice, we might have the answer turn out to depend on whether or not we accept the axiom of choice. We can tile the plane with hexagons as below Figure 2 to obtain a proper 7-coloring of the graph. The result is originally due to Hadwiger and Debrunner [8].

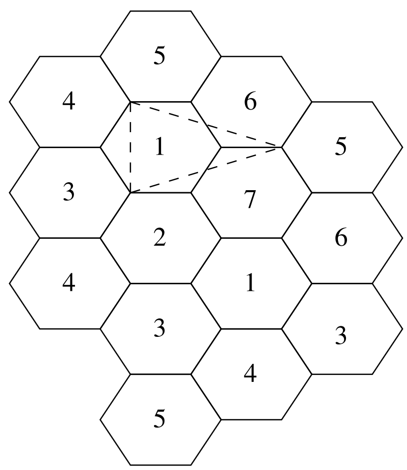

Figure 2.

For each point inside the hexagon, color that point with the number inside the hexagon. For each point on an edge or vertex, color it with the lowest color of the hexagons incident to it. If the side length of the hexagon is slightly less than one half, no two poins in or on the boundary of the single hexagon are at a distance one from each other. Also, the distance between any two hexagons of the same color is greater than one, so we have a proper seven coloring of the plane with seven colors. Hence $4 \leq \chi(G(R^2, \{1\})) \leq 7$.

We would also tile the plane with squares instead of hexagons, and obtain a proper 9-coloring of the plane. This can be extended to cubes in three dimensions, for a proper 27 coloring of R^3. We cannot extend this directly to arbitrary dimension because, in high dimension, the diagonal of the cube gets large compared to the side length.

	R^2	R^3	R^4	R^5	R^6	R^7	R^8
LB	4	6	7	9	11	15	16
UB	7	15	54				

In general, $(1.239 + o(1))^n \leq \chi(R^n) \leq (3 + o(1))^n$ [12].

4. Konig's Lemma and its Useful Consequences

Lemma 4.1 (Konig's Lemma)**.** *Suppose that (T, t_0) is a rooted tree at t_0 on $\mathcal{N}_0$-many vertices, and suppose that the degree of every vertex is finite. Then there is an infinite descending path in T starting at t_0.*

Proof. Let T_v denote the tree acquired by taking v and all of the paths that descend from v in our tree T. We create the required path by induction. Start at t_0. If we have made it to some vertex v. Let $v_1, \ldots, v_n$ be the descendents of v, and take as our inductive hypothesis the claim that T_v has infinitely many vertices in it. Then the trees $T_{v_1}, T_{v_2}, \ldots, T_{v_n}$ form a partition of the vertex set of $T_v - \{v\}$. As $T_v - \{v\}$ has infinite set of vertices, the pigeonhole principle tells us that one of these trees must contain infinitely many vertices. Let T_{v_i} be that tree and go to v_i. Repeating this process yields an infinite path descending through T, starting at t_0. The depth of the consequence of Konig's lemma is highly surprising. □

Corollary 4.1. *Suppose that G is a graph on $\mathcal{N}_0$-many vertices, such that any finite subgraph of G canbe k-colored. Then G can be k-colored.*

Proof. Create a tree as follows: Enumerate the vertices of G as $\{v_i\}_{i=1}^{\infty}$, and let the levels L_n of our tree be given by the collection of all k-colorings of $\{v_1, \ldots, v_n\}$. Draw an edge from any k-coloring of $\{v_1, \ldots, v_n\}$ to a coloring of $\{v_1, \ldots, v_{n+1}\}$ iff the coloring of $n+1$ vertices extends our coloring of n vertices. This is then a tree. It has infinitely many vertices, and the degree of any vertex is finite. This by Konig's Lemma, there's an infinite path. This path made of colorings that all agree with each other then gives a k-coloring of all of G.

The above motivates the following question: For countable graphs, to demonstrate k-colorability, it suffices to simply work on the collection of all finite graphs. Does the same hold for uncountable graphs? Specifically, to find the chromatic number of the plane, does it suffice to create an upper bound on all finite graphs embedded in the plane, with edges given by straight line segments of length 1? It turns out that the answer is yes. □

5. Limitation of Erdos-de Bruijn Result

One limitation of Erdos-de Bruijn result, has however remained a low key. They used Axioms of Choice, AC. So it is natural to ask, what if we have no choice? Absence of choice-in mathematics as in life may affect outcome.

Consider the following example of a distance graph on the real line R, whose χ depends upon the systems of axioms we choose for set theory. The example shows how the value of χ can be dramatically affected by the inclusion or the exclusion of the AC in the system of axioms for sets.

In 1904 Zermelo [15] formalized the AC that had previously been used informally.

Axioms of Choice (AC): Every family ϕ of non empty sets has a choice function, that is, there is a function f such that $f(s) \in S$ for every S from ϕ.

Many results in mathematics really need just a countable version of choice.

Countable AC (AC $\mathcal{N}_0$): Every countable family of non empty sets has a choice function.

In 1942, Bernays [2] introduced the following axiom.

Principle of Dependent Choices (DC): If E is a binary relation on a non empty set A, and for every $a \in A$ there exists $b \in A$ with aEb, then there exists a sequence $a_1, a_2, \ldots, a_n$ such that $a_n E a_{n+1}$ $\forall$ $n < w$.

AC implies DC (See Theorem 8.2 in [9] for example) but not conversely.

In turn, DC implies AC $\mathcal{N}_0$, but not conversely. DC is a weak form of AC and is sufficient for the classical theory of Lebegue measure. In particular, DC is sufficient for Falconer's result [6] formulated in question above.

We will make use of the following axiom:

Lebegue Measurable (LM): Every set of real numbers is Lebegue Measurable.
As always, ZF stands for Zermelo - Fraenkel system of axioms for sets, and ZFC = ZF + AC.

Assuming the existence of an inaccessible cardinal, Solovay constructed in 1964 (and published in 1970) a model that proved the following consistency result [13].

Theorem 5.1 (Solovay Theorem)**.** [13] *The system of Axioms ZF + DC + LM is consistent.*

As Jech [9] observes in the Solovay model, every set of reals differs from a Borel set by a set by a set of measure zero.

Finally we say a set $X \subseteq R$ has the Baire property if there is an open set U such that $X \Delta U$ (Symmetric difference) is meager (or of first category) i.e., a countable union of nowhere dense sets.

Example : We define a graph G as follows: the set R of real numbers serves as the vertex set, and the set of edges in $\{(s,t) : s - t - \sqrt{2} \in Q\}$.

Claim 1: In ZFC, $\chi(G) = 2$.

Let $S = \{q + n\sqrt{2} : q \in Q, n \in Z\}$. Define an equivalence relation E on R as : sEt iff $s - t \in S$. Let Y be a set of representatives for E. For $t \in R$, let $y(t) \in Y$ be such that $tEy(t)$. We define a 2-coloring $c(t)$ as follows: $c(t) = \ell$, $\ell = 0, 1$ iff there exists $n \in Z$ such that $t - y(t) - 2n\sqrt{2} - \ell\sqrt{2} \in Q$.

Without AC the chromatic situation changes dramatically.

Claim 2: In ZF + AC$\mathcal{N}_0$ + LM, $\chi(G) \neq$ any positive integer n nor even to $\mathcal{N}_0$.

The proof follows from the first of the following two statements.

1. If $A_1, A_2, \ldots, A_n, \ldots$ are measurable subsets of R and $\bigcup_{n<w} A_n \supseteq [0,1)$, then atleast one set A_n contains two adjacent vertices of the graph G.

2. If $A \subseteq [0,1)$ and A contains no pair of adjacent vertices of G then A is null (of Lebegue measure zero).

We can replace ZF + LM by ZF + "Every set of real numbers has the property of Baire". Is AC relevant to $\chi(R^2)$? The answer depends upon the value of χ which we do not know yet.

Conditional Theorem: Assume that any finite UDG (plane) has $\chi \ngtr 4$. Then in ZFC $\chi(R^2) = 4$, In ZF + DC + LM, $\chi(R^2)$ is 5, 6 or 7.

Proof. The above claim is true due to [3]. The system implies that every subset S of R^2 is Lebegue measurable. Indeed, S is measurable iff there is a Borel set B such that the symmetric difference $S \Delta B$ is null. Thus, every plane set differs from a Borel set by a null set. We can think of a unit segment $I = [0,1]$ as a set of infinite binary tractions and observe that the bijection $I \to I^2$ defined as $0.a_1a_2\ldots a_n \to (0.a_1a_3\ldots;0.a_2a_4\ldots)$ preserves null sets. Due to Falconer result [6], we can now conclude that $\chi(R^2) \geq 5$.

The problem of finding $\chi(R^2)$ has withstood all assaults in the general case, leaving us a wide range of X being 4, 5, 6 or 7 precisely because the answer depends on the system of axioms we choose for set theory. □

6. Role of Axiom of Choice

If, instead of $\chi(G)$, we work with "finite limit chromatic number" $\chi_{fin}(G)$, which is defined as the upper bound of $\chi(G_0)$ for all finite subgraphs G_0 of G, then we may not require axiom of choice. But the question of computing $\chi_{fin}(R^2)$ is precisely the same as that of computing $\chi(R^2)$ in the presence of axiom of choice. Observe that the statement $\chi_{fin}(R^2) = n$ is an arithmetical one (i.e., the one that can be stated in the language of first order arithmetic, viz., the one which states that every finite unit distance graph with real algebraic coordinates can be colored with n colors and atleast one requires this number of colors), its truth value does not depend on the axiom of choice. (because, the Godel constructible universe L has the same integers, so the same true arithmetical statements as the real universe V of set theory). So one might claim that right HNP in the absence of axiom of choice concerns the value of $\chi_{fin}(R^2)$, not $\chi(R^2)$ (which might be aritifically higher bcause certain colorings are not available in the absence of choice) the value of $\chi_{fin}(R^2)$ is a purely arithmetical question and therefore independent of set theoretic subtleties.

Note: The fields of characteristic 2 gets out of the ways because, if E is a field of char 2, then $(y_1-x_1)^2+(y_2-x_2)^2 = 1$ is equivalent to $(y_1-x_1)+(y_2-x_2) = 1$. That is, $\lambda(y-x) = 1$ where λ is the E-linear from $(z_1,z_2) \to z_1+z_2$. Complete 1 to be a basis of E as an F_2 vector space (this uses axiom of choice) and let $\widehat{\lambda}(z)$ be the coordinate on 1 of $\lambda(z)$. Then we get a coloring of E^2 with two colors if we choose the color of x according to the value of $\widehat{\lambda}(x) \in F_2$. As 1 color does not suffice, this shows that $\chi(E^2) = 2$.

7. Graph Dimension and Chromatic Number

By dim G, dimension of a graph G we mean the least integer n such that G can be embedded into R^n with every edge of G having length 1. We refer to such a embedding as unit-embedding. So if $dim\ G = n$, then there exists 1-1 mapping $f : V(G) \to R^n$, defined as $f(v_k) = (x_{k_1}, x_{k_2}, \ldots, x_{k_n})$ such that $v_kv_j \in E(G)$ whenever $\sum_{i=1}^{n}(x_{k_i} - x_{j_i})^2 = 1$.

The dim of a graph is related to many open problems in discrete geometry. The HNP, which seeks to determine the least number of colors needed to properly color the points of R^2 that are unit distance apart can be reduced to finding the largest possible chromatic number of a finite graph of dimension 2. Generalizations of HNP to higher dimensions n reduce to finding the largest possible chromatic number of a graph of dim n.

Since no systematic method for finding the dimension of an arbitrary graph is known, bounding the dimension of such graphs is an area of graph theory ripe for discoveries.

Note that we can deem platonic polyhedral graphs to be the 3-regular graphs which are the skeletons of platonic solids. The cube has dimension 2 and the tetrahedron, octahedron, dodecahedron and icosahedron have dim 3.

It is interesting to observe that $dim(K_n) = n-1$. To see this, note that a vertex set of a unit embedding of K_{r+1} is no more than a set of $(r+1)$ equidistant points. So it is enough to find a least positive integer m such that R^m contains such a set of $(r+1)$ points is when $m=r$. Let us assume that each such equidistance is unity and v_0 is the origin. Let $\overline{v}_1$, $\overline{v}_2$, $\ldots$, $\overline{v}_k$ be k vectors between v_0 and the other r equidistance points $v_1,\ldots,v_r$ in our set. As $\overline{v}_i\overline{v}_i = 1$ for all $1 \le i \le r$ and $\|\overline{v}_i - \overline{v}_j\| = 1$ for all $i \ne j$, we have $(\overline{v}_i.\overline{v}_i)^2 - 2(\overline{v}_i\overline{v}_j) + (\overline{v}_j.\overline{v}_j)^2 = 1$ and hence $\overline{v}_i,\overline{v}_j = \frac{1}{2}$. So one can construct the gram matrix $GM(\overline{v}_1,\ldots,\overline{v}_r)$ where $GM(\overline{v}_1,\ldots,\overline{v}_r) = \begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \cdots & \frac{1}{2} \\ \cdots & & \cdots & \\ \frac{1}{2} & \frac{1}{2} & \cdots & 1 \end{vmatrix}$. We know that this det is non-zero iff $\overline{v}_1,\ldots,\overline{v}_r$ are linearly independent. $GM(\overline{v}_1,\ldots,\overline{v}_r)$ has eigen value $\frac{1}{2}$ with multiplicity $r-1$. As trace is the sum of eigen values we get $tr(GM(\overline{v}_1,\ldots,\overline{v}_r)) = \sum_{i=1}^{r} \lambda_i = (r-1)\frac{1}{2} + \lambda_r$ showing $\lambda_r = \frac{(r+1)}{2}$. As the above matrix does not have 0 as eigen value its determinant is non-zero. So $\overline{v}_1,\ldots,\overline{v}_r$ are linearly independent. Since R^m can contain r linearly independent vectors iff $m \ge r$, we see that $r+1$ equidistant points $\overline{v}_1,\ldots,\overline{v}_r$ can be embedded in R^m iff $m \ge r$. So it follows that $dim\ K_{r+1} = r$ and hence $dim(K_n) = n-1$.

Also note that if G can be unit-embedded in $R^{dim\ G}$, then so is $H \subseteq G$. Then $dim\ G \ge dim\ H$. As $w(G)$ is the order of the greatest clique of G, we conclude that $dim(G) \ge w(G) - 1$. Then while partitioning G into χ-partite graph where each partite is of the same color we can deduce that $dim\ G \le 2\chi(G)$.

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