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Trigonometric Functional equations on monoids

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Abstract: Let M be a monoid and $\varphi : M \to M$ be an endomorphism (not necessarily involutive). In this paper, we find the solutions $f, g : M \to \mathbb{C}$ of each of the following functional equations

$$\begin{aligned} f(xy) - f(\varphi(y)x) &= 2f(x)g(y), \quad x, y \in M, \\ f(xy) + g(\varphi(y)x) &= 2f(x)g(y), \quad x, y \in M, \\ f(xy) + f(\varphi(y)x) &= 2g(x)g(y), \quad x, y \in M, \end{aligned}$$

in terms of multiplicative functions on M.

Key words: Functional equation, d'Alembert, monoid, endomorphism

1. Set up, notation and terminology

Throughout the paper we work in the following framework and with the following notation and terminology. We use it without explicit mentioning.

M is a monoid, that is a semigroup [a set equipped with an associative composition rule $(x, y) \mapsto xy$] with an identity element that we denote e. The map $\varphi : M \to M$ denotes a monoid endomorphism, i.e.,

- $\varphi(e) = e$;
- $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in M$.

A function $A : M \to \mathbb{C}$ is called additive, if it satisfies $A(xy) = A(x) + A(y)$ for all $x, y \in M$.

We say that a function $\chi : M \to \mathbb{C}$ is multiplicative, if $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in M$. It is well known that if M is a group, then every non-zero multiplicative function on M is a character of M, i.e., a homomorphism from M into the multiplicative group of non-zero complex numbers.

By $\mathcal{N}(M, \varphi)$ we mean the vector space of the solutions $\theta : M \to \mathbb{C}$ of the homogeneous equation

$$\theta(xy) - \theta(\varphi(y)x) = 0, \quad x, y \in M.$$

If M is a topological space, then we let $C(M)$ denote the algebra of continuous functions from M into $\mathbb{C}$.

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2. Introduction

In the papers [2–4] about vibrating strings d'Alembert studied not just what is now called d'Alembert's functional equation, i.e.,

$$g(x+y)+g(x-y)=2g(x)g(y), \quad x,y\in\mathbb{R}, \tag{1}$$

in which $g:\mathbb{R}\to\mathbb{R}$ is the unknown function, but also the functional equation

$$f(x+y)-f(x-y)=g(x)h(y), \quad x,y\in\mathbb{R}, \tag{2}$$

in which $f,g,h:\mathbb{R}\to\mathbb{R}$ are unknown functions. For further contextual and historical discussion we refer, e.g., to [1, 5, 6, 10].

Let S be a semigroup, and let $\sigma\in Hom(S,S)$ satisfy $\sigma^2=id$. In [12], Stetkær introduced and solved the following generalization of d'Alembert's functional equation (1)

$$g(xy)+g(\sigma(y)x)=2g(x)g(y), \quad x,y\in S, \tag{3}$$

where $g:S\to\mathbb{C}$ is the unknown function. He named it a variant of d'Alembert's functional equation. In [5], Ebanks and Stetkær determined the complex-valued solutions (f,g,h) of the functional equation

$$f(xy)-f(\sigma(y)x)=g(x)h(y), \quad x,y\in S, \tag{4}$$

when S is a group (or a monoid generated by its squares). This equation is a natural extension of the functional equation (2). Two important specials cases of (4) are

$$f(xy)-f(\sigma(y)x)=2f(x)g(y), \quad x,y\in S, \tag{5}$$

$$f(xy)-f(\sigma(y)x)=2g(x)g(y), \quad x,y\in S. \tag{6}$$

Eq. (5), in the case where S is an abelian group and $\sigma=-id$, has been solved in [14, Theorem 12.10] by methods from spectral analysis.

In [8], Fadli et al. studied the solutions $g:S\to\mathbb{C}$ of the functional equation

$$g(xy)+g(\phi(y)x)=2g(x)g(y), \quad x,y\in S, \tag{7}$$

where $\phi:S\to S$ is an endomorphism (i.e. $\phi(xy)=\phi(x)\phi(y)$ for all $x,y\in S$). This equation is a generalization of the variant (3) of d'Alembert's functional equation since ϕ need not be involutive. They showed that any solution $g:S\to\mathbb{C}$ of (7) can be expressed in terms of multiplicative functions on S (see [8, Theorem 3.1]).

In [9], Sabour determined the complex-valued solutions (f,g) of the functional equation

$$f(xy)+f(\phi(y)x)=2f(x)g(y), \quad x,y\in S,$$

when S is a group (or a monoid generated by its squares). This equation, in the case where $\phi^2=id$, has been solved by Fadli et al. in [7].

The present paper studies similar functional equations, i.e., functional equations with an endomorphism. It gives extensions of each of the functional equations (5) and (7). More precisely, we solve the functional equations

$$f(xy)-f(\varphi(y)x) = 2f(x)g(y), \quad x,y\in M, \tag{8}$$

$$f(xy)+g(\varphi(y)x) = 2f(x)g(y), \quad x,y\in M, \tag{9}$$

$$f(xy)+f(\varphi(y)x) = 2g(x)g(y), \quad x,y\in M, \tag{10}$$

by expressing their solutions in terms of multiplicative functions on M. Thus this work is a continuation of the earlier works [8] and [9] about d'Alembert's and Wilson's functional equations with an endomorphism.

Our new contributions to the theory are the following. First, the underlying space M is just a monoid, not necessarily a group. Second, the involutive automorphism σ found in the literature is in the functional equations studied in this paper replaced by a monoid endomorphism.

3. Solution of equation (8)

We begin by proving the following two lemmas which will serve to prove our main result in this section (Theorem 3.1).

Lemma 3.1. *Let S be a semigroup and let $\chi : S \to \mathbb{C}$ be a multiplicative function such that $2\chi \circ \varphi - \chi$ is multiplicative. Then $\chi \circ \varphi = \chi$.*

Proof. The function $\chi_1 := 2\chi \circ \varphi - \chi$ is multiplicative by assumption. Rewriting this definition as

$$\chi \circ \varphi + \chi \circ \varphi = \chi + \chi_1.$$

From [10, Corollary 3.19], we get that $\chi \circ \varphi = \chi$. □

Lemma 3.2. *Let the pair $f, g : M \to \mathbb{C}$ be a solution of (8).*

(i) For all $x, y \in M$, we have

$$f(xy) - f(\varphi(xy)) = 2f(x)g(y) + 2f(\varphi(y))g(x). \tag{11}$$

(ii) If $f, g \neq 0$, then $f(e) \neq 0$.

Proof. (i) Replacing x by $\varphi(y)$ and y by x in (8) we get that

$$f(\varphi(y)x) - f(\varphi(xy)) = 2f(\varphi(y))g(x). \tag{12}$$

Summing this last equation and (8), we get (11).

(ii) Suppose that $f(e) = 0$. Then (8) with $x = e$ gives

$$f = f \circ \varphi.$$

By using this equality in (11), we arrive at

$$f(x)g(y) + f(y)g(x) = 0.$$

This implies that $f = 0$ or $g = 0$ (see [10, Exercise 1.1 (b)]) which is absurd. So $f(e) \neq 0$. □

The following theorem solves the functional equation (8) on an arbitrary monoid. For the notation $\mathcal{N}(M, \varphi)$ see Section 1.

Theorem 3.1. *The pair $f, g : M \to \mathbb{C}$ satisfies (8) if and only if it has one the following forms.*

(a) $f = 0$ and g is arbitrary.

(b) $g = 0$ and $f \in \mathcal{N}(M, \varphi)$.

(c) *There exist a non-zero multiplicative function* $\chi : M \to \mathbb{C}$ *with* $\chi \neq \chi \circ \varphi$ *and a constant* $\alpha \in \mathbb{C}\backslash\{0\}$ *such that*

$$f = \alpha\chi \quad and \quad g = \frac{\chi - \chi \circ \varphi}{2}.$$

In this case $f \neq 0$ *and* $g \neq 0$.

Moreover, if M *is a topological monoid,* $f, g \neq 0$, *and* $g \in C(M)$, *then* $f, \chi, \chi \circ \varphi \in C(M)$.

Proof. If $f = 0$ (resp. $g = 0$) we deal with Case (a) (resp. Case (b)). So during the rest of the proof we will assume that $f \neq 0$ and $g \neq 0$.

Putting $x = e$ in (8) we get that

$$f - f \circ \varphi = 2f(e)g. \tag{13}$$

So equality (11) becomes

$$f(e)g(xy) = f(x)g(y) + f(\varphi(y))g(x).$$

Using (13) and Lemma 3.2 (ii) we can reformulate this last equation as follows

$$g(xy) = g(x)[\frac{f(y)}{f(e)} - g(y)] + g(y)[\frac{f(x)}{f(e)} - g(x)].$$

This shows that the pair (g, h), where $h := \frac{f}{f(e)} - g$, satisfies the sine addition law. Since $g \neq 0$, then Lemma 3.4 in [5] tells us that there exist two multiplicative functions $\chi_1, \chi_2 : M \to \mathbb{C}$ such that

$$h = \frac{\chi_1 + \chi_2}{2},$$

and we have only the following two cases.

Case 1: Assume that $\chi_1 \neq \chi_2$. In that case $g = c(\chi_1 - \chi_2)$ for some constant $c \in \mathbb{C}\backslash\{0\}$ and

$$\begin{aligned} f &= f(e)[g + h] = f(e)[c(\chi_1 - \chi_2) + \frac{1}{2}(\chi_1 + \chi_2)] \\ &= f(e)[(\frac{1}{2} + c)\chi_1 + (\frac{1}{2} - c)\chi_2]. \end{aligned}$$

Subcase 1.1: If $c = -\frac{1}{2}$, then $f = f(e)\chi_2$ and hence $\chi_2 \neq 0$ because $f \neq 0$. Substituting f into (8) we find that $\chi_1 = \chi_2 \circ \varphi$ and arrive at the solution in case (c) with $\chi = \chi_2$ and $\alpha = f(e)$.

Subcase 1.2: If $c = \frac{1}{2}$, then $f = f(e)\chi_1$ and hence $\chi_1 \neq 0$ because $f \neq 0$. Substituting f into (8) we get that $\chi_2 = \chi_1 \circ \varphi$ and arrive at the solution in case (c) with $\chi = \chi_1$ and $\alpha = f(e)$.

Subcase 1.3: If $c \neq \pm\frac{1}{2}$, then $f = c_1\chi_1 + c_2\chi_2$ where $c_1 = f(e)(\frac{1}{2} + c) \neq 0$ and $c_2 = f(e)(\frac{1}{2} - c) \neq 0$. Substituting f into (8) we find after a reduction that

$$\begin{aligned} &c_1\chi_1(x)[(1 - 2c)\chi_1(y) - \chi_1 \circ \varphi(y) + 2c\chi_2(y)] + c_2\chi_2(x)[(1 + 2c)\chi_2(y) \\ &\quad -\chi_2 \circ \varphi(y) - 2c\chi_1(y)] = 0 \end{aligned}$$

for all $x, y \in M$. Since $\chi_1 \neq \chi_2$ we get from the theory of multiplicative functions (see for instance [10, Theorem 3.18(d)]) that both terms are 0, so

$$\begin{cases} \chi_1(x)[(1-2c)\chi_1(y) - \chi_1 \circ \varphi(y) + 2c\chi_2(y)] = 0 \\ \chi_2(x)[(1+2c)\chi_2(y) - \chi_2 \circ \varphi(y) - 2c\chi_1(y)] = 0 \end{cases} \tag{14}$$

for all $x, y \in M$. Since $\chi_1 \neq \chi_2$ at least one of χ_1 and χ_2 is not zero. Using (14), we get that

$$(1-2c)\chi_1 + 2c\chi_2 = \chi_1 \circ \varphi + 0 \quad \text{or} \quad (1+2c)\chi_2 - 2c\chi_1 = \chi_2 \circ \varphi + 0. \tag{15}$$

By applying [13, Proposition A.2] to (15) we get that

$$[(1-2c)\chi_1 = \chi_1 \circ \varphi \ \text{ and } \ \chi_2 = 0] \quad \text{or} \quad [\chi_1 = 0 \ \text{ and } \ (1+2c)\chi_2 = \chi_2 \circ \varphi].$$

In both cases we get $c = 0$, because $\varphi(e) = e$. This subcase does not apply, because $c \neq 0$ by assumption. This completes Case 1.

Case 2: Assume that $\chi_1 = \chi_2 = \chi$. Since M is a monoid, then $\chi \neq 0$ (see [5, Lemma 3.4]). Furthermore, we have

$$g(xy) = g(x)\chi(y) + g(y)\chi(x) \quad \text{for all} \quad x, y \in M \tag{16}$$

and $f = f(e)[g + h] = f(e)[g + \chi]$. Using (13), we arrive at

$$g + g \circ \varphi = \chi - \chi \circ \varphi. \tag{17}$$

Now, substituting f into (8) and using (16), we get after a reduction that

$$g(x)[\chi(y) - \chi \circ \varphi(y) - 2g(y)] + \chi(x)[\chi(y) - \chi \circ \varphi(y)] = \chi(x)[g(y) + g \circ \varphi(y)]$$

for all $x, y \in M$. Which by using (17) becomes

$$g(x)[\chi(y) - \chi \circ \varphi(y) - 2g(y)] = 0 \quad \text{for all} \quad x, y \in M.$$

Since $g \neq 0$, we infer that $g = \frac{\chi - \chi \circ \varphi}{2}$. Replacing this in (17) we find that $\chi \circ \varphi^2 = 2\chi \circ \varphi - \chi$. So $2\chi \circ \varphi - \chi$ is multiplicative. Therefore $\chi \circ \varphi = \chi$ (see Lemma 3.1) which implies that $g = 0$, absurd. This finishes the necessity assertion.

Conversely, simple computations prove that the formulas above for (f, g) define solutions of (8).

The continuity statements follow from [10, Theorem 3.18(d)]. □

As an immediate consequence of Theorem 3.1, we have the following result.

Corollary 3.1. *The only complex-valued solution of the functional equation*

$$f(xy) - f(\varphi(y)x) = 2f(x)f(y), \quad x, y \in M,$$

is $f \equiv 0$.

In the following corollary, we give a simple example of equation (8) where our monoid is the group of real numbers under addition.

Corollary 3.2. *Let $z_0 \in \mathbb{R} \setminus \{1\}$ be a fixed element. The continuous solutions $f, g : \mathbb{R} \to \mathbb{C}$ of the functional equation*

$$f(x+y) - f(x+z_0 y) = 2f(x)g(y), \quad x, y \in \mathbb{R},$$

are:

(a) $f = 0$ *and* g *is arbitrary in* $C(\mathbb{R})$.

(b) f *is a non-zero complex constant and* $g = 0$.

(c) $f(x) = \alpha e^{\lambda x}$ *and*

$$g(x) = \frac{e^{\lambda x} - e^{\lambda x z_0}}{2}, \quad x \in \mathbb{R},$$

for some non-zero complex numbers α, λ.

Proof. We apply Theorem 3.1 with $M = (\mathbb{R}, +)$ and $\varphi(x) = z_0 x$ for all $x \in \mathbb{R}$.

Let $\theta \in \mathcal{N}(M, \varphi)$. Then $\theta(x+y) = \theta(x + z_0 y)$ for all $x, y \in \mathbb{R}$, i.e., $\theta(x) = \theta(x + (z_0 - 1)y)$ for all $x, y \in \mathbb{R}$. Since $\mathbb{R}$ is divisible by $z_0 - 1$, because $z_0 \neq 1$, then θ is a constant.

Let $\chi \neq 0$ be a continuous multiplicative function on $\mathbb{R}$. Since $\mathbb{R}$ is a group, χ is a character, so $\chi(x) = e^{\lambda x}$ for some $\lambda \in \mathbb{C}$ (see for instance [10, Example 3.7(a)]). Thus $\chi(z_0 x) = \chi(x)$ for all $x \in \mathbb{R}$ if and only if $\lambda = 0$. This finishes the proof. □

4. Solution of equation (9)

The following theorem solves the functional equation (9), i.e.,

$$f(xy) + g(\varphi(y)x) = 2f(x)g(y), \quad x, y \in M,$$

on an arbitrary monoid.

Theorem 4.1. *The pair $f, g : M \to \mathbb{C}$ satisfies (9) if and only if it has one of the following two forms.*

(a) There exists a multiplicative function $\chi : M \to \mathbb{C}$ with $\chi \circ \varphi^2 = \chi$ such that

$$g = f = \frac{\chi + \chi \circ \varphi}{2}.$$

(b) There exists a multiplicative function $\chi : M \to \mathbb{C}$ with $\chi \circ \varphi = \chi$ and a constant $\alpha \in \mathbb{C} \setminus \{-1, 0, 1\}$ such that

$$g = \alpha f = \frac{1+\alpha}{2}\chi.$$

Moreover, if M is a topological monoid and $g \in C(M)$, then $f, \chi, \chi \circ \varphi \in C(M)$.

Proof. Checking that the stated functions satisfy (9) is done by elementary calculations, that we leave out. So, it is left to show that any solution $f, g : M \to \mathbb{C}$ of (9) falls into one of the indicated two forms.

Letting $y = e$ in (9), we obtain

$$f + g = 2g(e)f.$$

Case 1: Suppose $g(e) = 0$. Then $g = -f$ and hence the functional equation (9) becomes

$$(-f)(xy) - (-f)(\varphi(y)x) = 2(-f)(x)(-f)(y), \quad x, y \in M.$$

From Corollary 3.1, we see that $f = 0$ and hence $g = -f = 0$. This is case (a) above with $\chi = 0$.

Case 2: Suppose $g(e) = \frac{1}{2}$. Then $g = 0$ and hence the functional equation (9) becomes

$$f(xy) = 0 \quad \text{for all } x, y \in M.$$

Which implies that $f = 0$. Also this is case (a) above with $\chi = 0$.

Case 3: Suppose $g(e) = 1$. Then $g = f$ and hence g is a solution of the variant of d'Alembert's functional equation with an endomorphism

$$g(xy) + g(\varphi(y)x) = 2g(x)g(y), \quad x, y \in M, \tag{18}$$

which was solved by Fadli et al. on semigroups in [8]. According to [8, Theorem 3.1], there are only the following two possibilities:

(i) There exists a non-zero multiplicative function $\chi : M \to \mathbb{C}$ with $\chi \circ \varphi = 0$ such that $g = \frac{1}{2}\chi$. This does not apply. Indeed, using [10, Lemma A.28] and the fact that $e = \varphi(e)$ we see that

$$\chi \neq 0 \Rightarrow \chi(e) = 1 \Rightarrow \chi \circ \varphi(e) = 1 \Rightarrow \chi \circ \varphi \neq 0.$$

(ii) There exists a multiplicative function $\chi : M \to \mathbb{C}$ with $\chi \circ \varphi^2 = \chi$ such that $g = (\chi + \chi \circ \varphi)/2$. This is case (a) of our statement.

Case 4: Finally we suppose $g(e) \notin \{0, \frac{1}{2}, 1\}$. Then $g = \alpha f$, where $\alpha = 2g(e) - 1 \notin \{-1, 0, 1\}$. Using (9), we see that g satisfies the functional equation

$$g(xy) + \alpha g(\varphi(y)x) = 2g(x)g(y), \quad x, y \in M. \tag{19}$$

Putting here $x = e$, we obtain $g + \alpha g \circ \varphi = 2g(e)g$, which implies that $g = g \circ \varphi$ because $\alpha = 2g(e) - 1 \neq 0$.

Next letting $\varphi(y)$ for x and x for y in (19) and using $g = g \circ \varphi$, we get

$$g(\varphi(y)x) + \alpha g(xy) = 2g(x)g(y), \quad x, y \in M.$$

Multiplying this last equation by α and subtracting it from (19), we obtain

$$(1 - \alpha^2)g(xy) = 2(1 - \alpha)g(x)g(y), \quad x, y \in M. \tag{20}$$

Since $\alpha \neq \pm 1$, Eq. (20) becomes

$$g(xy) = \frac{2}{1 + \alpha} g(x)g(y), \quad x, y \in M.$$

Which means that the function $\chi := \frac{2}{1+\alpha} g$ is multiplicative. From g being even with respect to φ we see that $\chi \circ \varphi = \chi$. This is case (b) of our statement.

The continuity statements follow from [10, Theorem 3.18 (d)]. □

In the following corollary, we determine all solutions of a special case of equation (9) on the real line.

Corollary 4.1. *Let $z_0 \in \mathbb{R}\setminus\{-1,1\}$ be a fixed element. The solutions $f,g:\mathbb{R}\to\mathbb{C}$ of the functional equation*

$$f(x+y)+g(x+z_0y)=2f(x)g(y), \quad x,y\in\mathbb{R}, \tag{21}$$

are:

(a) $g=f=0$.

(b) $g=f=1$.

(c) $g=\alpha f=\frac{1+\alpha}{2}$, *where* $\alpha\in\mathbb{C}\backslash\{-1,0,1\}$.

Proof. We apply Theorem 4.1 with $M=(\mathbb{R},+)$ and $\varphi(x)=z_0x$ for all $x\in\mathbb{R}$.

Let $\chi\neq 0$ be a multiplicative function on $\mathbb{R}$. Since $\mathbb{R}$ is a group, χ is a character, so $\chi(x)\neq 0$ for all $x\in\mathbb{R}$. If $\chi(z_0^2x)=\chi(x)$ for all $x\in\mathbb{R}$, then $\chi((z_0^2-1)x)=1$ for all $x\in\mathbb{R}$. But $(z_0^2-1)\mathbb{R}=\mathbb{R}$, because $z_0\neq\pm1$, so we get that $\chi=1$. Hence the only solutions of (21) are those stated in Corollary 4.1. □

5. Solution of equation (10)

In this section, we solve the functional equation (10), i.e.,

$$f(xy)+f(\varphi(y)x)=2g(x)g(y), \quad x,y\in M,$$

by expressing its solutions in terms of multiplicative functions.

Theorem 5.1. *The pair $f,g:M\to\mathbb{C}$ satisfies (10) if and only if there exist a multiplicative function $\chi:M\to\mathbb{C}$ with $\chi\circ\varphi^2=\chi$ and a constant $\alpha\in\mathbb{C}\backslash\{0\}$ such that*

$$g=\frac{\alpha}{2}(\chi+\chi\circ\varphi) \quad \text{and} \quad f=\alpha g.$$

Moreover, if M is a topological monoid and $g\in C(M)$, then $f,\chi,\chi\circ\varphi\in C(M)$.

Proof. Letting $y=e$ in (10), we obtain $f=g(e)g$.

Case 1: Suppose $g(e)=0$. Then $f=0$ and hence the functional equation (10) becomes

$$g(x)g(y)=0 \quad \text{for all } x,y\in M.$$

Which implies that $g=0$. Then the pair (f,g) has the desired form with $\chi=0$.

Case 2: Suppose $g(e)\neq 0$. Replacing f in (10), we get

$$g(e)[g(xy)+g(\varphi(y)x)]=2g(x)g(y), \quad x,y\in M.$$

Which means that the function $\frac{g}{g(e)}$ satisfies the variant (18) of d'Alembert's functional equation with an endomorphism. So that there exists a multiplicative function $\chi:M\to\mathbb{C}$ with $\chi\circ\varphi^2=\chi$ such that $g=\frac{g(e)}{2}(\chi+\chi\circ\varphi)$. So (f,g) has the desired form with $\alpha=g(e)$.

The other direction of the proof is trivial to verify.

The continuity statements follow from [10, Theorem 3.18 (d)]. □

In view of Theorem 5.1, we obtain the following result.

Corollary 5.1. *Let* $z_0 \in \mathbb{R} \setminus \{-1, 1\}$ *be a fixed element. The only solutions* $f, g : \mathbb{R} \to \mathbb{C}$ *of the functional equation*

$$f(x+y) + f(x+z_0y) = 2g(x)g(y), \quad x, y \in \mathbb{R},$$

are $g = \alpha$ *and* $f = \alpha^2$*, where* $\alpha \in \mathbb{C}$*.*

Proof. As the proof of Corollary 4.1 with the needed corrections. □

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