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On *GF*-projective modules and dimension

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Abstract

In this paper, we introduce the notion of (strongly) *GF*-projective modules. We show that a module is projective if and only if it is *GF*-projective and its Gorenstein flat dimension is at most 1, if and only if it is strongly *GF*-projective and Gorenstein flat. Moreover, we investigate (global) *GF*-projective dimensions of modules and rings, and some applications are presented.

Keywords: Gorenstein flat module, *GF*-projective module, (global) *GF*-projective dimension.

Subject Classification: Primary 16E10; Secondary 16E30.

Introduction

Throughout this article, R is an associative ring with identity and all modules are unitary. Unless stated otherwise, an R-module will be understood to be a left R-module. As usual, $pd_R(M)$, $id_R(M)$ and $fd_R(M)$ will denote the projective, injective and flat dimensions of an R-module M, respectively and $l.gldim(R)$ will denote the left global dimension of ring R. For an R-module M, M^* stands for the character module of M. For unexplained concepts and notations, we refer the reader to [11].

Recall that a ring R is called Gorenstein if it is n-Gorenstein for some non-negative integer n (a ring R is called n-Gorenstein if it is a left and right Noetherian ring with self-injective dimension at most n on both sides for some non-negative integer n). Clearly, Gorenstein rings are natural generalizations of quasi-Frobenius rings (a ring R is called quasi-Frobenius if it is a left and right Noetherian ring and it is an injective left R-module). In the relative homological algebra, Gorenstein rings play an important role and non-commutative Gorenstein rings were defined and studied by Iwanaga in [8] and [9]. Later Enochs and Jenda defined and studied the so-called Gorenstein projective, Gorenstein injective and Gorenstein flat modules and developed Gorenstein homological algebra, see [2] for details.

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It is well-known that the projective, injective and flat dimensions of modules play an important role. In the 2004's, the closely related Gorenstein projective, Gorenstein injective and Gorenstein flat dimensions were given and studied by H. Holm [6]. The Gorenstein injective dimension, $Gid_R(M)$, of an R-module M is defined by declaring that $Gid_R(M) \leq n$ if and only if M has a Gorenstein injective resolution of length n. Similarly, one defines the Gorenstein projective dimension, $Gpd_R(M)$, and Gorenstein flat dimension, $Gfd_R(M)$, of M, respectively. To study the homological properties of Gorenstein homological algebra more precisely, in [3, 4], Z. Gao studied the Ext-orthogonal and Tor-orthogonal modules of Gorenstein injective modules, and in [12, 13], T. Zhao further studied the Ext-orthogonal modules of Ding projective [5] (or strongly Gorenstein flat [1]) and Ding injective [5] (or Gorenstein FP-injective [10]) modules. Inspired by this, we will introduce the concept of GF-projective modules in terms of the Ext-orthogonality of Gorenstein flat modules and discuss the GF-projective dimensions of modules and rings.

In Section 2, we give the concept of GF-projective modules, and present some of the general properties. We show that an R-module M is projective if and only if M is GF-projective and $Gfd_RM \leq 1$, if and only if M is strongly GF-projective and Gorenstein flat; moreover, $(\mathcal{SG}, \mathcal{SG}^{\perp})$ is a hereditary cotorsion theory, where $\mathcal{SG}$ is the class of strongly GF-projective modules. We also show that over an n-Gorenstein ring, every (strongly) GF-projective right R-module is (strongly) GI-flat and a finitely generated right R-module is (strongly) GI-flat if and only if it is (strongly) GF-projective. In Section 3, we give the notions of the GF-projective dimension, $l.GF\text{-}pd_R(M)$, of an R-module M and the global GF-projective dimension of a ring R, defined by $l.GFdim(R) = \sup\{l.GF\text{-}pd_R(M) | M \text{ is an } R\text{-module}\}$. Moreover, we define the so-called "global" dimension of rings R:

$$l.\text{GF-}ID(R) = \sup\{id_R(M) | M \text{ is Gorenstein flat } R\text{-module}\}.$$

and show that $l.GFdim(R) = l.\text{GF-}ID(R)$. In addition, other applications of those dimensions defined in this way are presented.

2. The GF-projective modules

We begin with the following.

Definition 2.1. An R-module M is called *GF-projective* if $\text{Ext}^1_R(M, N) = 0$ for any Gorenstein flat R-module N.

By the definition, we have the following result.

Lemma 2.2. *The class of GF-projective R-modules is closed under extensions, direct sums and direct summands.*

Clearly, every projective module is GF-projective, but the converse is not true in general as shown in the following.

Proposition 2.3. *An R-module M is projective if and only if M is GF-projective and $l.Gfd_R(M) \leq 1$.*

Proof. $\Rightarrow$ It is trivial.

$\Leftarrow$ Let M be a GF-projective R-module. Consider an exact sequence $0 \to K \to P \to M \to 0$ of R-modules with P projective. Because $l.Gfd_R(M) \leq 1$, K is Gorenstein flat, and so $\text{Ext}^1_R(M, K) = 0$. Thus $0 \to K \to P \to M \to 0$ is split. Therefore M is projective as a direct summand of P. □

Let $\mathcal{F}$ be a class of R-modules. By an *$\mathcal{F}$-preenvelope* of an R-module M we mean a morphism $\varphi : M \to F$ where $F \in \mathcal{F}$ such that for any morphism $f : M \to F^{'}$ with $F^{'} \in \mathcal{F}$, there is a morphism $g : F \to F^{'}$ such that $g \circ \varphi = f$. If furthermore, when $F^{'} = F$ and $f = \varphi$ the only such g are automorphisms of F, then $\varphi : M \to F$ is called an *$\mathcal{F}$-envelope* of M. Dually, we have the definition of $\mathcal{F}$-(pre)cover of an R-module. Note that $\mathcal{F}$-envelopes and $\mathcal{F}$-covers may not exist in general, but if they exist, they are unique up to isomorphism.

Next we give some characterizations of GF-projective R-modules.

Proposition 2.4. *Let M be an R-module. Then the following are equivalent:*

(1) *M is GF-projective;*

(2) *For every exact sequence $0 \longrightarrow L \xrightarrow{f} N \longrightarrow M \longrightarrow 0$, where N is Gorenstein flat, $f : L \to N$ is a Gorenstein flat preenvelope of L;*

(3) *M is a cokernel of a monic Gorenstein flat preenvelope $g : A \to B$ with B projective.*

(4) *The functor $\mathrm{Hom}_R(M, -)$ is exact with respect to each exact sequence*

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$$

with A Gorenstein flat.

Proof. (1)$\Rightarrow$(2) Assume M is GF-projective. For every exact sequence

$$0 \longrightarrow L \xrightarrow{f} N \longrightarrow M \longrightarrow 0\ ,$$

where N is Gorenstein flat, we have the following exact sequence

$$\mathrm{Hom}_R(N, F) \xrightarrow{f^*} \mathrm{Hom}_R(L, F) \longrightarrow \mathrm{Ext}^1_R(M, F)\ ,$$

where F is Gorenstein flat. By Definition 2.1, $\mathrm{Ext}^1_R(M, F) = 0$, and hence $\mathrm{Hom}_R(N, F) \to \mathrm{Hom}_R(L, F)$ is an epimorphism, that is, for any $h : L \to F$, there exists $h' : N \to F$ such that $f^*(h') = h'f = h$. Hence $f : L \to N$ is a Gorenstein flat preenvelope of L.

(2)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(1) By the hypothesis, there is an exact sequence $0 \longrightarrow A \xrightarrow{g} B \longrightarrow M \longrightarrow 0$. Then for any Gorenstein flat R-module N, we have the following exact sequence

$$\mathrm{Hom}_R(B, N) \xrightarrow{g^*} \mathrm{Hom}_R(A, N) \longrightarrow \mathrm{Ext}^1_R(M, N) \longrightarrow \mathrm{Ext}^1_R(B, N) = 0\ .$$

On the other hand, it is easy to verify that $\mathrm{Hom}_R(B, N) \longrightarrow \mathrm{Hom}_R(A, N) \longrightarrow 0$ is exact by (3). Thus $\mathrm{Ext}^1_R(M, N) = 0$ and hence M is GF-projective.

(1) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) For any Gorenstein flat R-module N, there exists an exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow L \longrightarrow 0$$

with E injective, which induces the following exact sequence

$$\mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(M, L) \longrightarrow \mathrm{Ext}^1_R(M, N) \longrightarrow \mathrm{Ext}^1_R(M, E) = 0\ .$$

By (4), $\mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(M, L) \longrightarrow 0$ is exact. Hence $\mathrm{Ext}^1_R(M, N) = 0$ for any Gorenstein flat R-module N. Thus M is GF-projective. $\square$

Recall from [2] that an R-module is called coreduced if it has no nonzero projective quotient modules.

Corollary 2.5. *For an exact sequence* $0 \longrightarrow L \overset{f}{\longrightarrow} P \longrightarrow M \longrightarrow 0$ *with* P *projective, if* f *is a Gorenstein flat envelope, then* M *is coreduced* GF*-projective.*

Proof. By Proposition 2.4, M is GF-projective, so it suffices to show that M is coreduced. Assume that Q is a projective quotient module of M, then $P = Q \oplus_R N$ for some R-module N. Let $p : P \to N$ is the projection and $i : N \to P$ the inclusion. Then $ipf = f$ since $f(L) \subseteq N$. Hence ip is an isomorphism since f is an envelope. This implies that i is epic and so $Q = 0$, which means that M is coreduced. □

Let M be an R-module. We will call M *strongly* GF*-projective* if $\mathrm{Ext}^i_R(M, N) = 0$ for any Gorenstein flat R-module N and any $i \geq 1$.

Lemma 2.6. *Let* M *be strongly* GF*-projective. Then*

(1) $\mathrm{Ext}^1_R(M, N) = 0$ *for any* R*-module* N *with* $l.Gfd_R(N) < \infty$.

(2) *either* M *is projective or* $l.Gfd(M) = \infty$.

Proof. (1) Since $l.Gfd_R(N) < \infty$, we may assume that $l.Gfd_R(N) = n < \infty$. Then we have the following exact sequence

$$0 \longrightarrow \widetilde{F}_n \longrightarrow \widetilde{F}_{n-1} \longrightarrow \cdots \longrightarrow \widetilde{F}_1 \longrightarrow \widetilde{F}_0 \longrightarrow N \longrightarrow 0$$

with each $\widetilde{F}_i$ Gorenstein flat. Let $L_{-1} = N$, $L_i = \mathrm{Im}(\widetilde{F}_i \to \widetilde{F}_{i-1})$ for $0 \leq i \leq n-1$ and $L_n = \widetilde{F}_n$. Then $0 \longrightarrow L_i \longrightarrow \widetilde{F}_i \longrightarrow L_{i-1} \longrightarrow 0$, $1 \leq i \leq n$, are exact. Hence we have

$$\mathrm{Ext}^1_R(M, N) \cong \mathrm{Ext}^2_R(M, L_0) \cong \cdots \cong \mathrm{Ext}^{n+1}_R(M, L_{n-1}) \cong \mathrm{Ext}^{n+2}_R(M, \widetilde{F}_n).$$

So we have $\mathrm{Ext}^1_R(M, N) = \mathrm{Ext}^{n+2}_R(M, \widetilde{F}_n) = 0$.

(2) Assume that $l.Gfd_R(M) < \infty$ and consider an exact sequence $0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$ with P projective. Since $l.Gfd_R(M) < \infty$, $l.Gfd_R(L) < \infty$. This implies that $\mathrm{Ext}^1_R(M, L) = 0$ by (1), so this sequence is split and hence M is projective. □

Corollary 2.7. *An* R*-module* M *is projective if and only if it is strongly* GF*-projective and Gorenstein flat.*

Proposition 2.8. *Let* R *be an* n*-Gorenstein ring and* M *an* R*-module. Then the following are equivalent:*

(1) M *is strongly* GF*-projective;*

(2) $\mathrm{Ext}^1_R(M, N) = 0$ *for any* R*-module* N*;*

(3) $\mathrm{Ext}^i_R(M, N) = 0$ *for any* R*-module* N *and any* $i \geq 1$.

Proof. (1)⇒(2) Note that $l.Gfd_R(M) \leq n$ since R is an n-Gorenstein ring by [2, Theorem 12.3.1]. By Lemma 2.6, this result holds.

(2)⇒(3) It is easy by the dimension shifting.

(3)⇒(1) By (3), M is projective. It follows from the fact that every projective module is strongly GF-projective. □

Lemma 2.9. *Let* M *be a* GF*-projective* R*-module. Then for every exact sequence* $0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$ *with* P *projective,* M *is strongly* GF*-projective if and only if* L *is strongly* GF*-projective.*

Proof. ⇒ It is easy.

$\Leftarrow$ Let L be strongly GF-projective. For every exact sequence $0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$ with P projective, we have the following exact sequence

$$0 = \operatorname{Ext}_R^{i-1}(L, F) \longrightarrow \operatorname{Ext}_R^i(M, F) \longrightarrow \operatorname{Ext}_R^i(P, F) = 0 \ , \quad i \geq 2$$

with F Gorenstein flat. So $\operatorname{Ext}_R^i(M, F) = 0$ for $i \geq 2$. Moreover, $\operatorname{Ext}_R^1(M, F) = 0$ by the hypothesis. Thus M is strongly GF-projective. $\square$

Given a class $\mathcal{C}$ of R-modules, we will denote by $\mathcal{C}^\perp = \{M \mid \operatorname{Ext}_R^1(C, M) = 0 \text{ for all } C \in \mathcal{C}\}$ the right orthogonal class of $\mathcal{C}$, and by ${}^\perp\mathcal{C} = \{M \mid \operatorname{Ext}_R^1(M, C) = 0 \text{ for all } C \in \mathcal{C}\}$ the left orthogonal class of $\mathcal{C}$. A pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a *cotorsion theory* if $\mathcal{F}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{F}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be *hereditary* if whenever $0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$ is exact with $L, L'' \in \mathcal{F}$, then L' is also in $\mathcal{F}$, or equivalently, with $L, L' \in \mathcal{C}$, then L'' is also in $\mathcal{C}$.

Proposition 2.10. *Let $\mathcal{SG}$ be the class of strongly GF-projective R-modules, then $(\mathcal{SG}, \mathcal{SG}^\perp)$ is a hereditary cotorsion theory.*

Proof. Clearly, $\mathcal{SG} \subseteq^\perp (\mathcal{SG}^\perp)$. It suffices to show ${}^\perp(\mathcal{SG}^\perp) \subseteq \mathcal{SG}$. Let $M \in \mathcal{SG}$ and $N \in \mathcal{SG}^\perp$, consider the exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ with P projective, we have $\operatorname{Ext}_R^2(M, N) \cong \operatorname{Ext}_R^1(K, N)$. Moreover, consider the exact sequence $0 \longrightarrow N \longrightarrow E \longrightarrow V \longrightarrow 0$ with E injective, then we have $\operatorname{Ext}_R^1(M, V) \cong \operatorname{Ext}_R^2(M, N) = 0$, and hence $V \in \mathcal{SG}^\perp$. Now let $L \in^\perp(\mathcal{SG}^\perp)$, then $\operatorname{Ext}_R^2(L, N) \cong \operatorname{Ext}_R^1(L, V) = 0$. By induction, we have $\operatorname{Ext}_R^i(L, N) = 0$ for all $i > 0$. In particular, $\operatorname{Ext}_R^i(L, G) = 0$ for all Gorenstein flat R-modules G and all $i > 0$. Thus $L \in \mathcal{SG}$.

Finally, $(\mathcal{SG}, \mathcal{SG}^\perp)$ is hereditary by Lemma 2.9. $\square$

Proposition 2.11. *Let M be an R-module. If $\operatorname{Ext}_R^i(M, N) = 0$ for any i with $1 \leq i \leq n+1$ and any Gorenstein flat R-module N, then every kth syzygy of M is GF-projective for $0 \leq k \leq n$.*

Proof. Let L_k be a k-th syzygy of M. Then we have the following exact sequence

$$0 \longrightarrow L_k \longrightarrow P_{k-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each P_i, $0 \leq i \leq k-1$, projective. For any Gorenstein flat R-module N, we have that $\operatorname{Ext}_R^1(L_k, N) \cong \operatorname{Ext}_R^{k+1}(M, N)$. Note that $\operatorname{Ext}_R^{k+1}(M, N) = 0$ by the hypothesis, and so $\operatorname{Ext}_R^1(L_k, N) = 0$, which means that L_k is GF-projective. $\square$

Recall that a right R-module M is called *GI-flat* [4] if $\operatorname{Tor}_1^R(M, N) = 0$ for any Gorenstein injective R-module N, and M is called *strongly GI-flat* [4] if $\operatorname{Tor}_i^R(M, N) = 0$ for any Gorenstein injective R-module N and any $i \geq 1$.

Next we discuss the relation between GF-projective and GI-flat modules.

Lemma 2.12. *Let R be an n-Gorenstein ring. Then every (strongly) GF-projective right R-module is (strongly) GI-flat.*

Proof. Let N be a Gorenstein injective left R-module. Since R is n-Gorenstein, by [2, Corollary 10.3.9], N^* is Gorenstein flat. Moreover, we have the following standard isomorphisms: $\operatorname{Ext}_R^i(M, N^*) \cong \operatorname{Tor}_i^R(M, N)^*$ for all $i \geq 1$. So the result holds. $\square$

Lemma 2.13. *Let R be a right noetherian ring. Then every finitely generated (strongly) GI-flat right R-module is (strongly) GF-projective.*

Proof. Let M be finitely generated. Since R is right noetherian, by [11, Theorem 10.66], we have an isomorphisms: $\mathrm{Ext}^i_R(M,N)^* \cong \mathrm{Tor}^R_i(M,N^*)$ for all $i \geq 1$. Let N be a Gorenstein flat right R-module, then N^* is Gorenstein injective by [2, Proposition 10.3.3]. So the result holds. □

Proposition 2.14. *Let R be a Gorenstein ring and M a finitely generated right R-module. Then M is (strongly) GI-flat if and only if M is (strongly) GF-projective.*

Proof. It follows immediately from Lemmas 2.12 and 2.13. □

We call a ring R *left Gorenstein perfect* if every Gorenstein flat R-module is projective.

Proposition 2.15. *The following are equivalent:*

(1) *R is left Gorenstein perfect;*
(2) *Every Gorenstein flat R-module is GF-projective;*
(3) *Every Gorenstein flat R-module is strongly GF-projective.*

Proof. (1)⇒(3)⇒(2) are clear.

(2)⇒(1) Let N be Gorenstein flat. Then we have an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ with P projective. Since N is Gorenstein flat, K is Gorenstein flat by [2, Theorem 10.3.14]. Moreover N is GF-projective by (2). Thus $\mathrm{Ext}^1_R(N,K) = 0$. Therefore $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ is split, which means that N is projective as a direct summand of P. □

3. The GF-projective dimension of modules and rings

We know that the homological dimension is a valuable tool in homological algebra and a number of well-known theorems can be reformulated in terms of the homological dimension. Therefore, it is interesting and valuable to study the homological dimension in details.

We begin with the following definition.

Definition 3.1. Let R be a ring. The *left GF-projective dimension*, $l.GF\text{-}pd_R(M)$, of an R-module M is defined to be the smallest non-negative integer n such that $\mathrm{Ext}^{n+1}_R(M,N) = 0$ for any Gorenstein flat left R-module N. The *left global GF-projective dimension*, $l.GFdim(R)$, of R is defined as

$$l.GFdim(R) = \sup\{l.GF\text{-}pd_R(M) | M \text{ is an } R\text{-module}\}.$$

Similarly, we can define $r.GFdim(R)$. If R is commutative, we drop r and l.

If M is strongly GF-projective, we set $l.GF\text{-}pd_R(M) = 0$.

Proposition 3.2. *Let M be an R-module. Then the following are equivalent:*

(1) $l.GF\text{-}pd_R(M) \leq n$;
(2) $\mathrm{Ext}^{n+i}_R(M,N) = 0$ *for all Gorenstein flat R-modules N and all $i \geq 1$;*
(3) *For every exact sequence*

$$0 \longrightarrow L_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0\ ,$$

where P_i, $0 \leq i \leq n-1$ are projective, then L_n is strongly GF-projective;

(4) *There exists an exact sequence*

$$0 \longrightarrow \widetilde{P}_n \longrightarrow \widetilde{P}_{n-1} \longrightarrow \cdots \longrightarrow \widetilde{P}_1 \longrightarrow \widetilde{P}_0 \longrightarrow M \longrightarrow 0$$

with each $\widetilde{P}_i$ *strongly* GF*-projective.*

Proof. (1)⇒(2) It is easy.

(2)⇒(3) For every exact sequence

$$0 \longrightarrow L_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each P_i, $0 \leq i \leq n-1$ projective. Let $L_0 = M$, $L_j = \mathrm{Im}(P_j \to P_{j-1})$ for $1 \leq j \leq n-1$, then $0 \longrightarrow L_j \longrightarrow P_j \longrightarrow L_{j-1} \longrightarrow 0$ are exact. So we have the following

$$\mathrm{Ext}^i_R(L_n, N) \cong \mathrm{Ext}^{i+1}_R(L_{n-1}, N) \cong \cdots \cong \mathrm{Ext}^{i+n}_R(M, N).$$

Thus $\mathrm{Ext}^i_R(L_n, N) \cong \mathrm{Ext}^{i+n}_R(M, N) = 0$ for any Gorenstein flat R-module N by (2), and so L_n is strongly GF-projective.

(3)⇒(4) It is trivial.

(4)⇒(1) For every exact sequence

$$0 \longrightarrow \widetilde{P}_n \longrightarrow \widetilde{P}_{n-1} \longrightarrow \cdots \longrightarrow \widetilde{P}_1 \longrightarrow \widetilde{P}_0 \longrightarrow M \longrightarrow 0$$

with each $\widetilde{P}_i$ strongly GF-projective. Let $K_1 = \mathrm{Ker}(\widetilde{P}_0 \to M)$, $K_{i+1} = \mathrm{Ker}(\widetilde{P}_i \to \widetilde{P}_{i-1})$ for $i \geq 1$. Since each $\widetilde{P}_i$ is strongly GF-projective, we have $\mathrm{Ext}^{n+1}_R(M, N) \cong \mathrm{Ext}^n_R(K_1, N) \cong \cdots \cong \mathrm{Ext}^1_R(\widetilde{P}_n, N) = 0$ where N is Gorenstein flat. So $l.GF\text{-}pd_R(M) \leq n$. □

Now we define the following so-called "global" dimension of rings.

Definition 3.3. Let R be a ring. We define

$$l.\mathrm{GF}\text{-}ID(R) = \sup\{id_R(M) | M \text{ is Gorenstein flat } R\text{-module}\}.$$

If $l.\mathrm{GF}\text{-}ID(R) = 0$, then we call that R is a *left* GFI *ring*.

Firstly, we have the following results.

Proposition 3.4. *Let* R *be a ring. Then*

$$l.GF\text{-}ID(R) = sup\{id_R(M) | M \text{ is an } R\text{-module with } l.Gfd_R(M) < \infty\}.$$

Proof. (1) It is clear that

$$l.\mathrm{GF}\text{-}ID(R) \leq \sup\{id_R(M) | M \text{ is an } R\text{-module with } l.Gfd_R(M) < \infty\}.$$

So we only need to show that

$$\sup\{id_R(M) | M \text{ is an } R\text{-module with } l.Gfd_R(M) < \infty\} \leq l.\mathrm{GF}\text{-}ID(R).$$

If $l.\mathrm{GF}\text{-}ID(R) = \infty$, then we have completed the proof. So we assume that $l.\mathrm{GF}\text{-}ID(R) < \infty$. For any R-module M with $l.Gfd_R(M) = n < \infty$, by the definition, there exists an exact sequence

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with each F_i Gorenstein flat. Clearly, $id_R(F_i) \leq l.\mathrm{GF}\text{-}ID(R)$. Let $L_0 = M$, $L_i = \mathrm{Im}(F_i \to F_{i-1})$, $L_n = F_n$, then for every short exact sequence $0 \longrightarrow L_i \longrightarrow F_i \longrightarrow L_{i-1} \longrightarrow 0$, we have the fact that

$id_R(L_i) \leq l.\text{GF-}ID(R)$ and $id_R(F_i) \leq l.\text{GF-}ID(R)$ imply $id_R(L_{i-1}) \leq l.\text{GF-}ID(R)$, and hence $id_R(M) \leq l.\text{GF-}ID(R)$, as desired. □

Next we give a relation of the left global GF-projective dimension $l.GFdim(R)$ and $l.\text{GF-}ID(R)$.

Proposition 3.5. *Let R be a ring. Then $l.GFdim(R) = l.\text{GF-}ID(R)$.*

Proof. We first show that $l.GFdim(R) \leq l.\text{GF-}ID(R)$. If $l.\text{GF-}ID(R) = \infty$, then we have completed the proof. So we assume that $l.\text{GF-}ID(R) = m < \infty$. Let M be an R-module, then $\text{Ext}_R^{m+1}(M, N) = 0$ for any Gorenstein flat R-module N since $id_R(N) \leq m$ by the hypothesis, and hence $l.GF\text{-}pd_R(M) \leq m$. This show that $l.GFdim(R) \leq m = l.\text{GF-}ID(R)$.

Now we show that $l.\text{GF-}ID(R) \leq l.GFdim(R)$. Indeed, assume that $l.GFdim(R) = n < \infty$. For any Gorenstein flat R-module N, let M be an R-module, it follows that $l.GF\text{-}pd_R(M) \leq n$, and so $\text{Ext}_R^{n+1}(M, N) = 0$, which induces that $id_R(N) \leq n$. Thus $l.\text{GF-}ID(R) \leq n = l.GFdim(R)$, as desired. □

Corollary 3.6. *Let R be a ring. Then R is a left GFI ring if and only if every R-module is GF-projective, if and only if every R-module is strongly GF-projective.*

Remark 3.7. *By the above, $l.GFdim(R)$ measures how far away a ring is from being left GFI.*

Corollary 3.8. *Let R be a ring. Then the following are equivalent:*

(1) *$l.GFdim(R) \leq 1$;*

(2) *$l.GF\text{-}ID(R) \leq 1$;*

(3) *Every submodule of a GF-projective R-module is GF-projective;*

(4) *Every submodule of a projective R-module is GF-projective.*

(5) *Every submodule of a strongly GF-projective R-module is strongly GF-projective;*

(6) *Every submodule of a projective R-module is strongly GF-projective.*

Proof. (1)⇔(2) It is trivial by Proposition 3.5.

(2)⇒(3) Let L be a submodule of a GF-projective R-module M. Then we have the following exact sequence

$$\text{Ext}_R^1(M, N) \longrightarrow \text{Ext}_R^1(L, N) \longrightarrow \text{Ext}_R^2(M/L, N)$$

for any Gorenstein flat R-module N. Moreover, $\text{Ext}_R^1(M, N) = 0$ since M is GF-projective and $\text{Ext}_R^2(M/L, N) = 0$ since $id_R(N) \leq 1$ by the hypothesis. Thus $\text{Ext}_R^1(L, N) = 0$, and hence L is GF-projective, as desired.

(3)⇒(4) It is trivial.

(4)⇒(1) For any left R-module M, consider an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ with P projective, we have the following exact sequence

$$\text{Ext}_R^1(K, N) \longrightarrow \text{Ext}_R^2(M, N) \longrightarrow \text{Ext}_R^2(P, N)$$

for any Gorenstein flat R-module N. Moreover, $\text{Ext}_R^1(K, N) = 0$ since K is GF-projective as a submodule of P by (4) and $\text{Ext}_R^2(P, N) = 0$. Thus $\text{Ext}_R^2(M, N) = 0$, and hence $l.GF\text{-}pd_R(M) \leq 1$. Therefore $l.GFdim(R) \leq 1$.

(5)⇒(6) It follows from the fact that every projective module is strongly GF-projective.

(6)⇒(5) Let M be a strongly GF-projective R-module and K a submodule of M. For the R-module M/K, we have an exact sequence $0 \longrightarrow K' \longrightarrow P \longrightarrow M/K \longrightarrow 0$ with P projective. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & K' & = & K' & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M/K & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

By (6), K' is strongly GF-projective as a submodule of P. Thus N is strongly GF-projective. Now for the exact sequence $0 \longrightarrow K \longrightarrow N \longrightarrow P \longrightarrow 0$ and any Gorenstein flat R-module G, we have the following exact sequence

$$0 = \mathrm{Ext}_R^i(N,G) \longrightarrow \mathrm{Ext}_R^i(K,G) \longrightarrow \mathrm{Ext}_R^{i+1}(P,G) = 0\ ,$$

and hence $\mathrm{Ext}_R^i(K,G) = 0$ for all $i \geq 1$. Therefore, K is strongly GF-projective, as desired.

(1)⇒(6) Let P be a projective R-module and K a submodule of P. For any Gorenstein flat R-module G, the exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow P/K \longrightarrow 0$ induces the following exact sequence

$$\mathrm{Ext}_R^i(P,G) \longrightarrow \mathrm{Ext}_R^i(K,G) \longrightarrow \mathrm{Ext}_R^{i+1}(P/K,G)$$

for all $i \geq 1$. Here, $\mathrm{Ext}_R^i(P,G) = 0$ since P is projective and $\mathrm{Ext}_R^{i+1}(P/K,G) = 0$ by (1). Therefore, $\mathrm{Ext}_R^i(K,G) = 0$ and hence K is strongly GF-projective.

(6)⇒(4) It is trivial. □

It is well known that the left global dimension $l.gldim(R) \leq 1$ of a left hereditary ring R. Now we give another characterization of hereditary rings as follows:

Corollary 3.9. *A ring R is left hereditary if and only if $l.GFdim(R) \leq 1$ and every GF-projective R-module is projective.*

Proof. If R is left hereditary, then every submodule of projective modules is projective. By Corollary 3.8, $l.GFdim(R) \leq 1$. So it suffices to show that every GF-projective R-module is projective. For any R-module N, consider an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ with P projective, we have the following exact sequence

$$\mathrm{Ext}_R^1(M,P) \longrightarrow \mathrm{Ext}_R^1(M,N) \longrightarrow \mathrm{Ext}_R^2(M,K)$$

for any GF-projective R-module M. Since R is left hereditary, $\mathrm{Ext}_R^2(M,K) = 0$ and clearly $\mathrm{Ext}_R^1(M,P) = 0$. Thus $\mathrm{Ext}_R^1(M,N) = 0$, and hence M is projective, as desired.

Conversely, for any R-module M, consider an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective. By Corollary 3.8, K is GF-projective, and so K is projective by the hypothesis. Thus $pd_R(M) \leq 1$ and hence R is hereditary. □

Theorem 3.10. *Let R be a commutative noetherian ring and n a non-negative integer. Then the following are equivalent:*

(1) $GFdim(R) \leq n$;

(2) For any Gorenstein flat R-module N and any flat R-module F, $id_R(N \otimes_R F) \leq n$.

Proof. (1)⇒(2) For any Gorenstein flat R-module N, since $GFdim(R) \leq n$, then $id_R(N) \leq n$. So we have an exact sequence

$$0 \longrightarrow N \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow 0$$

with each E^i injective. Then for any flat R-module F, we have the following exact sequence

$$0 \longrightarrow N \otimes_R F \longrightarrow E^1 \otimes_R F \longrightarrow \cdots \longrightarrow E^{n-1} \otimes_R F \longrightarrow E^n \otimes_R F \longrightarrow 0 .$$

Note that each $E^i \otimes_R F$ is injective by [7, Theorem 1.3], which induces that $id_R(N \otimes F) \leq n$.

(2)⇒(1) For any Gorenstein flat R-module N, we consider the following exact sequence

$$0 \longrightarrow N \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow L^n \longrightarrow 0$$

with each E^i injective. Then for any flat R-module F, we have the following exact sequence

$$0 \longrightarrow N \otimes_R F \longrightarrow E^1 \otimes_R F \longrightarrow \cdots \longrightarrow E^{n-1} \otimes_R F \longrightarrow L^n \otimes_R F \longrightarrow 0 .$$

Similarly, each $E^i \otimes_R F$ is injective by [7, Theorem 1.3]. But $id_R(N \otimes F) \leq n$, so $L^n \otimes_R F$ is injective. Thus L^n is injective by [7, Theorem 1.3] again, which induces that $id_R(N) \leq n$. Therefore $GFdim(R) \leq n$. □

Corollary 3.11. *Let R be a commutative noetherian ring. Then R is a GFI ring if and only if $N \otimes_R F$ is injective for any Gorenstein flat R-module N and any flat R-module F.*

We conclude this paper with the following results. Let

$$l.FID(R) = \sup\{id_R(M) \mid M \text{ is any flat } R\text{-module}\}.$$

Proposition 3.12. *Let R be a ring. Then*

$$l.FID(R) \leq l.GFdim(R) \leq l.gldim(R).$$

Proof. By Proposition 3.5, $l.GFdim(R) \leq l.gldim(R)$. So it suffices to prove $l.FID(R) \leq l.GFdim(R)$. Assume that $l.GFdim(R) = n < \infty$. Let F be a flat R-module with $id_R(F) = m < \infty$. We claim that $m \leq n$. Otherwise, let $m > n$. Let N be any R-module, then there exists an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$ with P projective, which induces the following exact sequence

$$\mathrm{Ext}_R^m(P, F) \longrightarrow \mathrm{Ext}_R^m(K, F) \longrightarrow \mathrm{Ext}_R^{m+1}(N, F) .$$

Note that $\mathrm{Ext}_R^m(P, F) = 0$ since P is projective and $\mathrm{Ext}_R^{m+1}(N, F) = 0$ since $id_R(F) = m$. So $\mathrm{Ext}_R^m(K, F) = 0$, which induces $id_R(F) < m$. This is a contradiction. So $m \leq n$ and hence $l.FID(R) \leq l.GFdim(R)$. □

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