Generalized Hyers-Ulam type stability of the additive functional equation inequalities with $2^n$-variables on an approximate group and ring homomorphism

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Abstract:
In this paper we study to solve additive functional inequality with $2^n$-variables and their Hyers-Ulam type stability. First are investigated results with a direction method of group homomorphism and last are investigated in ring homomorphism. Then I will show that the solutions of inequality are additive mapping. These are the main results of this paper.

Key words: stability, functional equation Banach space; generalized Hyers-Ulam stability. Jordan-homomorphism, Lie-homomorphism, equation functional inequality
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1. Introduction
The study of the functional equation stability originated from a question of S.M. Ulam [23], concerning the stability of group homomorphisms. Let $(G, \ast)$ be a group and let $(G', \circ, d)$ be a metric group with metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies

$$d\left(f(x \ast y), f(x) \circ f(y)\right) < \delta$$

for all $x, y \in G$ then there is a homomorphism $h : G \to G'$ with

$$d\left(f(x), h(x)\right) < \epsilon$$

for all $x \in G$, if the answer, is affirmative, we would say that equation of homomorphism $h(x \ast y) = h(y) \circ h(y)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation? Hyers[13] gave a first affirmative answer to the question of Ulam as follows:

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**Theorem 1.1.** (D. H. Hyers) Let \( \epsilon \geq 0 \) and let \( f \) be a function defined on an Abelian group \( (\mathbb{G}, +) \) with values in Banach spaces \( (\mathbb{Y}, +) \) satisfying
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon,
\]
for all \( x, y \in \mathbb{G} \) and some \( \epsilon \geq 0 \). Then there exists a unique additive mapping \( T : \mathbb{G} \to \mathbb{Y} \), such that
\[
\|f(x) - T(x)\| \leq \epsilon, \forall x \in \mathbb{G}.
\]

Next Th. M. Rassias [20] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded:

**Theorem 1.2.** (Th. M. Rassias.) Consider \( \mathbb{E}, \mathbb{E}' \) to be two Banach spaces, and let \( f : \mathbb{E} \to \mathbb{E}' \) be a mapping such that \( f(tx) \) is continous in \( t \) for each fixed \( x \). Assume that there exist \( \theta > 0 \) and \( p \in [0, 1] \) such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p), \forall x, y \in \mathbb{E}.
\]
then there exists a unique linear \( L : \mathbb{E} \to \mathbb{E}' \) satifies
\[
\|f(x) - L(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p, x \in \mathbb{E}.
\]

Next Badora in [6] provided the following result concerning the stability of a ring homomorphism:

**Theorem 1.3.** Let \( \mathbb{R} \) be a ring and \( \mathbb{Y} \) be Banach algebra and \( \epsilon, \delta \geq 0 \). Assume that \( f : \mathbb{R} \to \mathbb{Y} \) satisfies
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon
\]
and
\[
\|f(x \cdot y) - f(x)f(y)\| \leq \delta,
\]
\( \forall x, y \in \mathbb{R} \). Then there exists a unique ring homomorphism \( T : \mathbb{R} \to \mathbb{Y} \) such that
\[
\|f(x) - T(x)\| \leq \epsilon, \forall x \in \mathbb{R}.
\]

The stability problems for several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. Such as in 1978, Rassias in [20] prove a generalization of Hyers’ theorem the Cauchy difference to be unbounded and R. Badora in [6] prove generalization the result on a ring homomorphisms next in [24] proved the generalized Hyers’ theorem and...
Badora’ theorem. Recently, in [6, 13, 20, 24] the authors studied the Hyers-Ulam stability for the following functional inequalities

\[ \left\| f\left( \sum_{k=1}^{r} x_k \right) - \sum_{k=1}^{r} f(x_k) \right\|_Y \leq \epsilon, \forall \epsilon \geq 0. \]

and

\[ \left\| f\left( \prod_{j=1}^{r} x_j \right) - \prod_{j=1}^{r} f(x_j) \right\|_Y \leq \delta, \forall \delta \geq 0 \]

in group and ring homomorphisms. So that we solve and proved the Hyers-Ulam type stability for functional inequalities

\[ \left\| f\left( \sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=1}^{n} x_{n+j} \right) - \sum_{j=1}^{n} f(x_j) - \sum_{j=1}^{n} f\left( \frac{x_{n+j}}{n} \right) \right\|_Y \leq \epsilon, \forall \epsilon \geq 0 \] (1)

and

\[ \left\| f\left( \prod_{j=1}^{n} x_j + \frac{1}{n} \prod_{j=1}^{n} x_{n+j} \right) - \prod_{j=1}^{n} f(x_j) - \prod_{j=1}^{n} f\left( \frac{x_{n+j}}{n} \right) \right\|_Y \leq \delta, \forall \delta \geq 0 \] (2)

ie the functional inequalities with 2n-variables. Under suitable assumptions on spaces \( X \) and \( Y \), we will prove that the mappings satisfying the functional inequalities (1) and (2). Thus, the results in this paper are generalization of those in [6, 13, 20, 24] for functional inequalities with 2n-variables.

The paper is organized as followns: In section preliminaries we remind some basic notations in [3, 4, 14] such as solutions of the inequalities.

Section 3 is devoted to prove the Hyers-Ulam type stability of the additive functional inequalities (1) when \( X \) be an Abelian group and \( Y \) be a Banach space.

Section 4 is devoted to prove the Hyers-Ulam type stability of the additive functional inequality (1) and (2) when \( X \) be a ring and \( Y \) be a Banach algebra, \( X \) be an Abelian group and \( Y \) be a Banach space.

2. preliminaries


**Definition 2.1.** Let \( \{x_n\} \) be a sequence in a normed space \( X \).

1. A sequence \( \{x_n\}_{n=1}^{\infty} \) in a space \( X \) is a Cauchy sequence iff the sequence \( \{x_{n+1} - x_n\}_{n=1}^{\infty} \) converges to zero;

2. The sequence \( \{x_n\}_{n=1}^{\infty} \) is said to be convergent if, there exists \( x \in X \) such that, for any \( \epsilon > 0 \), there is a positive integer \( N \) such that
\[ \|x_n - x\| \leq \epsilon, \forall n \geq N. \]

Then the point \( x \in X \) is called the limit of sequence \( x_n \) and denoted by \( \lim_{n \to \infty} x_n = x \);

3. If every sequence Cauchy in \( X \) converges, then the normed space \( X \) is called a Banach space.

### 2.2. Solutions of the inequalities.

The functional equation

\[ f(x + y) = f(x) + f(y) \]

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an **additive mapping**.

### 3. Stability of Approximate Group Homomorphisms

Now, we first study the solutions of (1). Note that for this inequality, \( X \) be an Abelian group and \( Y \) is a Banach space. Under this setting, we can show that the mapping satisfying (1) is additive. These results are given in the following.

**Theorem 3.1.** Let \( X \) be an Abelian group and \( Y \) be Banach space. If \( \epsilon \geq 0, n \in \mathbb{N}, n \geq 2 \) and \( f : X \to Y \) such that

\[ \left\| f\left(\sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=1}^{n} x_{n+j}\right) - \sum_{j=1}^{n} f(x_j) - \sum_{j=1}^{n} f\left(x_{n+j} / n\right)\right\|_Y \leq \epsilon \tag{3} \]

for all \( x_1, x_2, \ldots, x_{2n} \in X \), then there exists a unique additive mapping \( H : X \to Y \) such that

\[ \left\| f(x) - H(x)\right\|_Y \leq \frac{1}{2n-1} \epsilon, \forall x \in X. \tag{4} \]

**Proof.** We will show that

\[ \left\| \frac{f((2n)^k x)}{(2n)^k} - f(x)\right\|_Y \leq \frac{1}{2n} \epsilon, \sum_{m=1}^{k} (2n)^{-m}, \forall x \in X. \tag{5} \]

for any positive integer \( k \) and for all \( x \in X \). The proof of (5) follows by induction on \( k \). With \( k = 1 \) and letting \( x_j = x, x_{n+j} = nx \) for all \( j = 1, 2, \ldots, n \) by the hypothesis (3), we have

\[ \left\| f(2nx) / 2n - f(x)\right\|_Y = \frac{1}{2n} \left\| f(2nx) - 2nf(x)\right\|_Y \leq \frac{1}{2n} \epsilon, \forall x \in X. \]

Assume now that (5) holds for \( k \) and we want to prove it for the case \( k + 1 \). Replacing \( x \) by \( 2nx \) in (5) we obtain

\[ \left\| \frac{f((2n)^k \cdot 2nx)}{(2n)^k} - f(2nx)\right\|_Y \leq \epsilon, \sum_{m=1}^{k} (2n)^{-m}, \forall x \in X. \]
therefore
\[ \left\| \frac{f(2n^{k+1}x)}{(2n)^{k+1}} - \frac{1}{2n} f(2nx) \right\|_Y \leq \epsilon \sum_{m=2}^{k+1} (2n)^{-m}, \forall x \in \mathbb{X}. \]

Now, using the triangle inequality we deduce
\[
\left\| \frac{f(2n^{k+1}x)}{(2n)^{k+1}} - f(x) \right\|_Y \leq \left\| \frac{f(2n^{k+1}x)}{(2n)^{k+1}} - \frac{1}{2n} f(2nx) \right\|_Y + \left\| \frac{1}{2n} f(2nx) - f(x) \right\|_Y \\
\leq \frac{\epsilon}{2n} + \epsilon \sum_{m=2}^{k+1} (2n)^{-m} \\
\leq \epsilon \sum_{m=1}^{k+1} (2n)^{-m}.
\]

Thus, (5) is valid for all \( k \in \mathbb{N} \). Since \( \sum_{m=1}^{k} (2n)^{-m} \) is increasingly convergent to \( \frac{1}{2n-1} \), we get from (5) that
\[
\left\| \frac{f(2n^{k+1}x)}{(2n)^{k+1}} - f(x) \right\|_Y \leq \frac{1}{2n-1} \epsilon, \forall x \in \mathbb{X}.
\]

(6)

Fixing an \( x \in \mathbb{X} \), for all \( h,k \in \mathbb{N} \) with \( h > k \), we have, from (6) that
\[
\left\| \frac{f(2n^{h}x)}{(2n)^{h}} - \frac{1}{(2n)^k} f(2n^k x) \right\|_Y = \frac{1}{(2n)^h} \left\| \frac{1}{(2n)^{h-k}} f(2n^h x) - f(2n^k x) \right\|_Y \\
\leq \frac{1}{(2n)^h} \cdot \frac{1}{2n-1} \epsilon.
\]

Therefore
\[
\lim_{h,k \to \infty} \left\| \frac{f(2n^{h}x)}{(2n)^{h}} - \frac{1}{(2n)^k} f(2n^k x) \right\|_Y = 0.
\]

Since \( Y \) is Banach space, the sequence \( \left\{ \frac{f(2n^{k}x)}{(2n)^k} \right\} \) converges. Set
\[
H(x) = \lim_{k \to \infty} \frac{f(2n^{k}x)}{(2n)^k}, \forall x \in \mathbb{X}.
\]

(7)

Then we obtain a mapping \( H : \mathbb{X} \to \mathbb{Y} \). From (10), for all \( x_1, x_2, \ldots, x_{2n} \in \mathbb{X} \) and for all \( k \in \mathbb{N} \), We compute that
\[ \left\| f \left( (2n)^k \left( \sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=1}^{n} x_{n+j} \right) \right) - \sum_{j=1}^{n} f \left( (2n)^k x_j \right) - \sum_{j=1}^{n} f \left( (2n)^k \frac{x_{n+j}}{n} \right) \right\|_Y \leq \epsilon, \]

and so
\[ \frac{1}{(2n)^k} \left\| f \left( (2n)^k \left( \sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=1}^{n} x_{n+j} \right) \right) - \sum_{j=1}^{n} f \left( (2n)^k x_j \right) - \sum_{j=1}^{n} f \left( (2n)^k \frac{x_{n+j}}{n} \right) \right\|_Y \leq \frac{1}{(2n)^k} \epsilon. \]

We will prove that \( H \) is additive. Consequently,
\[
\left\| H \left( \sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=1}^{n} x_{n+j} \right) - \sum_{j=1}^{n} H \left( x_j \right) - \sum_{j=1}^{n} H \left( \frac{x_{n+j}}{n} \right) \right\|_Y \\
= \lim_{k \to \infty} \left\| \frac{1}{(2n)^k} f \left( (2n)^k \left( \sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=1}^{n} x_{n+j} \right) \right) \right. \\
- \sum_{j=1}^{n} \frac{1}{(2n)^k} f \left( (2n)^k x_j \right) - \sum_{j=1}^{n} \frac{1}{(2n)^k} f \left( (2n)^k \frac{x_{n+j}}{n} \right) \left\|_Y \right. \\
\leq \lim_{k \to \infty} \left\| \frac{1}{(2n)^k} \epsilon \right\| = 0.
\]

It follows from (7) that
\[ \left\| H \left( \sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=1}^{n} x_{n+j} \right) - \sum_{j=1}^{n} H \left( x_j \right) - \sum_{j=1}^{n} H \left( \frac{x_{n+j}}{n} \right) \right\|_Y = 0. \]

Hence
\[ H \left( \sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=1}^{n} x_{n+j} \right) = \sum_{j=1}^{n} H \left( x_j \right) + \sum_{j=1}^{n} H \left( \frac{x_{n+j}}{n} \right), \]

for all \( x_1, x_2, ..., x_{2n} \in X \).

Clearly, \( H \left( 0 \right) = 0 \) and so \( H \) is an additive mapping. From (6) and (7) we see that (4) is valid. Now we prove the uniqueness of \( H \). Assume that \( H_1 : X \to Y \) is an additive mapping such that
\[ \left\| f \left( x \right) - H_1 \left( x \right) \right\| \leq \frac{1}{2n-1} \epsilon, \forall x \in X. \]

Since both \( H \) and \( H_1 \) are additive, we deduce that, for each \( \forall x \in X \) and for all \( n \in \mathbb{N} \),
\[ 2n \| H(x) - H_1(x) \|_Y = \| H(2nx) + H_1(2nx) \|_Y \]
\[ \leq \| H(2nx) - f(2nx) \|_Y + \| f(2nx) + H_1(2nx) \|_Y \]
\[ \leq \frac{2\epsilon}{2n-1}, \]
so that
\[ \| H(x) - H_1(x) \|_Y \leq \frac{2\epsilon}{n(2n-1)} \]
for all \( x \in \mathbb{X} \) and hence \( H(x) = H_1(x) \) for all \( x \in \mathbb{X} \). This completes the proof. \( \square \)

**Corollary 3.1.** Let \( \mathbb{X} \) be an Abelian group and \( \mathbb{Y} \) be Banach space. If \( \epsilon \geq 0 \), \( n \in \mathbb{N} \), \( n \geq 2 \), \( f(0) = 0 \) and \( f : \mathbb{X} \rightarrow \mathbb{Y} \) such that
\[ \| f \left( \sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=1}^{n} x_{n+j} \right) - \sum_{j=1}^{n} f(x_j) - \sum_{j=1}^{n} f \left( \frac{x_{n+j}}{n} \right) \|_Y \leq \epsilon, \] (8)
for all \( x_1, x_2, \ldots, x_{2n} \in \mathbb{X} \), then there exists a unique additive group homomorphism \( H : \mathbb{X} \rightarrow \mathbb{Y} \) such that
\[ \| f(x) - H(x) \|_Y \leq \frac{1}{2n-1} \epsilon, \forall x \in \mathbb{X}. \] (9)

**4. Stability of a Ring Homomorphism**

Now, we first study the solutions of (2). Note that for this inequality, \( \mathbb{X} \) be a ring and \( \mathbb{Y} \) is a Banach algebra and \( \mathbb{X} \) be an Abelian group and \( \mathbb{Y} \) is a Banach spaces. Under this setting, we can show that the mapping satisfying (2) is additive. These results are give in the following.

**Theorem 4.1.** Let \( \mathbb{R} \) be a ring and \( \mathbb{Y} \) be Banach algebra and \( \epsilon, \delta \geq 0 \) and \( n \in \mathbb{N}, \ n \geq 2 \). If a mapping \( f : \mathbb{R} \rightarrow \mathbb{Y} \) satisfies
\[ \| f \left( \sum_{j=1}^{n} x_j + \frac{1}{n} \sum_{j=1}^{n} x_{n+j} \right) - \sum_{j=1}^{n} f(x_j) - \sum_{j=1}^{n} f \left( \frac{x_{n+j}}{n} \right) \|_Y \leq \epsilon \] (10)
and
\[ \| f \left( \prod_{j=1}^{n} x_j + \frac{1}{n} \prod_{j=1}^{n} x_{n+j} \right) - \prod_{j=1}^{n} f(x_j) - \prod_{j=1}^{n} f \left( \frac{x_{n+j}}{n} \right) \|_Y \leq \delta \] (11)
for all \( x_1, x_2, \ldots, x_{2n} \in \mathbb{R} \), then there exists a unique additive mapping \( H : \mathbb{R} \rightarrow \mathbb{Y} \) such that
\[ H \left( \prod_{j=1}^{n} x_j + \frac{1}{n} \prod_{j=1}^{n} x_{n+j} \right) = \prod_{j=1}^{n} H(x_j) + \prod_{j=1}^{n} H \left( \frac{x_{n+j}}{n} \right) \] (12)
for all \(x_1, x_2, \ldots, x_{2n} \in \mathbb{R}\) and

\[
\left\| f(x) - H(x) \right\|_\nu \leq \frac{1}{2n-1} \epsilon, \forall x \in \mathbb{R}.
\] (13)

**Proof.** Theorem 3.1 show that there exists a unique additive mapping \(H: \mathbb{R} \to \mathbb{Y}\) satisfies (13). By the proof of Theorem 3.1, we see that the mapping \(H\) is give by

\[
H(x) = \lim_{k \to \infty} \frac{1}{(2n)^k} f \left( (2n)^k x \right), \forall x \in \mathbb{R}
\] (14)

for all \(x_1, x_2, \ldots, x_{2n} \in \mathbb{R}\), let

\[
h(x_1, x_2, \ldots, x_{2n}) = f \left( n \prod_{j=1}^{n} x_j + \frac{1}{n} \prod_{j=1}^{n} x_{n+j} \right) - \prod_{j=1}^{n} f(x_j) - \prod_{j=1}^{n} f \left( \frac{x_{n+j}}{n} \right).
\]

The using inequality (10), we get

\[
\lim_{k \to \infty} \frac{1}{(2n)^k} h \left( (2n)^k x_1, x_2, \ldots, x_{2n} \right) = 0.
\]

Therefore

\[
H \left( \prod_{j=1}^{n} x_j + \frac{1}{n} \prod_{j=1}^{n} x_{n+j} \right) = H \left( x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n} \right)
\]

\[
= \lim_{k \to \infty} \frac{1}{(2n)^k} f \left( (2n)^k \left( x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n} \right) \right)
\]

\[
= \lim_{k \to \infty} \frac{1}{(2n)^k} h \left( \left( (2n)^k x_1 \right) x_2 \cdots x_n + \frac{1}{n} \left( (2n)^k x_{n+1} \right) x_{n+2} \cdots x_{2n} \right)
\]

\[
= \lim_{k \to \infty} \frac{1}{(2n)^k} \left[ h \left( (2n)^k x_1, x_2, \ldots, x_{2n} \right) + f \left( (2n)^k x_1 \right) f(x_2) \cdots f(x_n)ight.
\]

\[
+ f \left( \frac{(2n)^k}{n} x_{n+1} \right) f \left( \frac{x_{n+2}}{n} \right) \cdots f \left( \frac{x_{2n}}{n} \right) \left]
\]

\[
= \prod_{j=1}^{n} H(x_j) + \prod_{j=1}^{n} H \left( \frac{x_{n+j}}{n} \right).
\]

\(\forall x_1, x_2, \ldots, x_{2n} \in \mathbb{R}\).

From the last equation and the additivity of \(H\) we see that, for all \(k \in \mathbb{N}\)
\[ H(x_1) f((2n)^k x_2) \cdots f(x_n) + H\left(\frac{x_{n+1}}{n}\right) f\left(\frac{(2n)^k \cdot x_{n+2}}{n}\right) \cdots f\left(\frac{x_{2n}}{n}\right) \]

\[ = H\left(x_1 \cdot (2n)^k x_2 \cdots x_n + \frac{1}{n} x_{n+1} \cdot (2n)^k x_{n+2} \cdots x_{2n}\right) \]

\[ = H\left((2n)^k \cdot x_1 \cdot x_2 \cdots x_n + \frac{1}{n} (2k)^k x_{n+1} \cdot x_{n+2} \cdots x_{2n}\right) \]

\[ = (2n)^k H\left(x_1\right) f\left(x_2\right) \cdots f\left(x_n\right) \]

\[ + (2n)^k H\left(\frac{x_{n+1}}{n}\right) f\left(\frac{x_{n+2}}{n}\right) \cdots f\left(\frac{x_{2n}}{n}\right) \]

and so

\[ H\left(x_1\right) f\left((2n)^k x_2\right) \cdots f\left(x_n\right) + H\left(\frac{x_{n+1}}{n}\right) f\left(\frac{(2n)^k \cdot x_{n+2}}{n}\right) \cdots f\left(\frac{x_{2n}}{n}\right) \]

\[ = H\left(x_1\right) f\left(x_2\right) \cdots f\left(x_n\right) + H\left(\frac{x_{n+1}}{n}\right) f\left(\frac{x_{n+2}}{n}\right) \cdots f\left(\frac{x_{2n}}{n}\right) . \]

Sending \( k \) to infinity, we see that

\[ H\left(x_1\right) H\left(x_2\right) H\left(x_3\right) \cdots f\left(x_n\right) + H\left(\frac{x_{n+1}}{n}\right) H\left(\frac{x_{n+2}}{n}\right) H\left(\frac{x_{n+3}}{n}\right) \cdots f\left(\frac{x_{2n}}{n}\right) \]

\[ = H\left(x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n}\right) , \quad (15) \]

\( \forall x_1, x_2, \ldots, x_{2n} \in \mathbb{R} \).

Suppose that

\[ H\left(x_1\right) H\left(x_2\right) H\left(x_3\right) \cdots H\left(x_{n-1}\right) f\left(x_n\right) + H\left(\frac{x_{n+1}}{n}\right) H\left(\frac{x_{n+2}}{n}\right) H\left(\frac{x_{n+3}}{n}\right) \]

\[ \cdots H\left(\frac{x_{2n-1}}{n}\right) f\left(\frac{x_{2n}}{n}\right) \]

\[ = H\left(x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n}\right) , \quad (16) \]
∀x₁, x₂, ..., x₂n ∈ ℜ. Then, from (16), we get that, for all k ∈ N.

\[
\frac{1}{(2n)^k} H(x_1) H(x_2) H(x_3) \cdots H(x_{n-1}) f((2n)^k x_n)
\]

\[\] + \frac{1}{(2n)^k} H\left(\frac{x_{n+1}}{n}\right) H\left(\frac{x_{n+2}}{n}\right) \cdots H\left(\frac{x_{2n-1}}{n}\right) f\left((2n)^k \cdot \frac{x_{2n}}{n}\right)

\[=\]

\frac{1}{(2n)^k} H\left((2n)^k \left(x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n}\right)\right)

\[=\]

\frac{1}{(2n)^k} H\left(x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n}\right)

(17)

By letting k → ∞ we see that

\[
H\left(x_1 x_2 \cdots x_n + \frac{1}{n} x_{n+1} x_{n+2} \cdots x_{2n}\right)
\]

(18)

∀x₁, x₂, ..., x₂n ∈ ℜ which is the desired identity (12)

Theorem 4.2. Let X be an Abelian group and Y be Banach space. If ϵ ≥ 0, ∀k ∈ N, and ϕ : X × X × X → X such that

\[
2^k \varphi(x, y, z) = \varphi(2^k x, y, z) = \varphi(x, 2^k y, z) = \varphi(x, y, 2^k z)
\]

∀k ∈ N, x, y, z ∈ X

and ψ : Y × Y × Y → Y be a continuous mapping such that

\[
2^k \psi(x, y, z) = \psi(2^k x, y, z) = \psi(x, 2^k y, z) = \psi(x, y, 2^k z)
\]

∀k ∈ N, x, y, z ∈ Y and ϵ, δ ≥ 0.

If f : X → Y satisfies

\[
\left\| f\left(\frac{x + y}{2} + z\right) - f\left(\frac{x + y}{2}\right) - f(z) \right\|_Y \leq \epsilon
\]

(19)

for all x, y, z ∈ X and

\[
\left\| f\left(\varphi(x, y, z)\right) - \psi\left(f(x), f(y), f(z)\right)\right\|_Y \leq \delta
\]

(20)

for all x, y, z ∈ X

Then there exists a unique additive mapping H : X → Y such that

\[
H\left(\varphi(x, y, z)\right) = \psi\left(H(x), H(y), H(z)\right)
\]

(21)
for all \( x, y, z \in X \)

\[
\|f(x) - H(x)\|_Y \leq \frac{1}{2} \epsilon, \forall x \in X.
\]  

(22)

Proof. We will show that

\[
\left\| \frac{f(2^k x)}{2^k} - f(x) \right\|_Y \leq \epsilon \sum_{m=1}^{k} 2^{-m}, \forall x \in X.
\]  

(23)

for any positive integer \( k \) and for all \( x \in X \). The proof of (23) follows by induction on \( k \). With \( k = 1 \) and letting \( x = y = z \) by the hypothesis (19), we have

\[
\left\| \frac{f(2x)}{2} - f(x) \right\|_Y = \frac{1}{2} \left\| f(2x) - 2f(x) \right\|_Y \leq \frac{1}{2} \epsilon, \forall x \in X.
\]

Assume now that (23) holds for \( k \) and we want to prove it for the case \( k + 1 \). Replacing \( x \) by \( 2x \) in (23) we obtain

\[
\left\| \frac{f(2^{k+1} x)}{2^{k+1}} - \frac{1}{2} f(2x) \right\|_Y \leq \epsilon \sum_{m=2}^{k+1} 2^{-m}, \forall x \in X.
\]

Now, using the triangle inequality we deduce

\[
\left\| \frac{f(2^{k+1} x)}{2^{k+1}} - f(x) \right\|_Y \leq \left\| \frac{f(2^{k+1} x)}{2^{k+1}} - \frac{1}{2} f(2x) \right\|_Y + \left\| \frac{1}{2} f(2x) - f(x) \right\|_Y
\]

\[
\leq \frac{\epsilon}{2} + \epsilon \sum_{m=1}^{k+1} 2^{-m}
\]

\[
\leq \epsilon \sum_{m=1}^{k+1} 2^{-m}.
\]

Thus, (23) is valid for all \( k \in \mathbb{N} \). Since \( \sum_{m=1}^{k+1} 2^{-m} \) is increasingly convergent to \( \frac{1}{2} \), we get from (23) that

\[
\left\| \frac{f(2^{k+1} x)}{2^{k+1}} - f(x) \right\|_Y \leq \frac{1}{2} \epsilon, \forall x \in X.
\]  

(24)

Fixing an \( x \in X \), for all \( h, k \in \mathbb{N} \) with \( h > k \), we have, from (24) that
\[ \left\| \frac{f(2^h x)}{2^h} - \frac{1}{2^h} f(2^k x) \right\|_Y = \frac{1}{2^h} \left\| \frac{1}{2^{h-k}} f(2^h x) - f(2^k x) \right\|_Y \]
\[ \leq \frac{1}{2^h} \frac{1}{2} \epsilon. \]

Therefore
\[ \lim_{h,k \to \infty} \left\| \frac{f(2^h x)}{2^h} - \frac{1}{2^h} f(2^k x) \right\|_Y = 0. \]

Since \( Y \) is Banach space, the sequence \( \left\{ f\left( \frac{2^k x}{2^k} \right) \right\} \) converges. Set
\[ H(x) = \lim_{k \to \infty} \frac{f(2^k x)}{2^k}, \forall x \in X. \quad (25) \]

Then we obtain a mapping \( H : X \to Y \). From (19), for all \( x, y, z \in X \) and for all \( k \in \mathbb{N} \), We compute that
\[ \left\| f\left( 2^k \left( \frac{x+y}{2} + z \right) \right) - f(2^k \frac{x+y}{2}) - f(2^k z) \right\|_Y \leq \epsilon, \]
and so
\[ \frac{1}{2^k} \left\| f\left( 2^k \left( \frac{x+y}{2} + z \right) \right) - f(2^k \frac{x+y}{2}) - f(2^k z) \right\|_Y \leq \frac{1}{2^k} \epsilon. \]

We will prove that \( H \) is additive. Consequently,
\[ \left\| H\left( \frac{x+y}{2} + z \right) - H\left( \frac{x+y}{2} \right) - H(z) \right\|_Y \]
\[ = \lim_{k \to \infty} \left\| \frac{1}{2^k} f\left( 2^k \left( \frac{x+y}{2} + z \right) \right) - \frac{1}{2^k} f\left( 2^k \frac{x+y}{2} \right) - \frac{1}{2^k} f(2^k z) \right\|_Y \]
\[ \leq \lim_{k \to \infty} \left\| \frac{1}{2^k} \epsilon \right\| = 0. \]

It follows from (25) that
\[ \left\| H\left( \frac{x+y}{2} + z \right) - H\left( \frac{x+y}{2} \right) - H(z) \right\|_Y = 0. \]

Hence
\[ H\left( \frac{x+y}{2} + z \right) = H\left( \frac{x+y}{2} \right) + H(z) \]
for all $x, y, z \in X$.

Clearly, $H(0) = 0$ and so $H$ is an additive mapping. From (24) and (25) we see that (22) is valid. Now we prove the uniqueness of $H$. Assume that $H_1 : X \to Y$ is an additive mapping such that

$$\|f(x) - H_1(x)\|_Y \leq \frac{1}{2} \epsilon, \forall x \in X.$$ 

Since both $H$ and $H_1$ are additive, we deduce that, for each $\forall x \in X$ and for all $k \in \mathbb{N}$,

$$k \left\| H(x) - H_1(x) \right\|_Y = \left\| H(kx) + H_1(kx) \right\|_Y \leq \left\| H(kx) - f(kx) \right\|_Y + \left\| f(kx) + H_1(kx) \right\|_Y \leq \epsilon,$$

so that

$$\left\| H(x) - H_1(x) \right\|_Y \leq \frac{\epsilon}{k}$$

for all $x \in X$ and hence $H(x) = H_1(x)$ for all $x \in X$.

Next to show that the mapping $H$ satisfies (20), we define

$$Q(x, y, z) = f\left(\varphi(x, y, z)\right) - \psi\left(f(x), f(y), f(z)\right), \forall x, y, z \in X.$$

Then from condition (20), we see that

$$\lim_{k \to \infty} \frac{1}{2^k} Q\left(2^k x, y, z\right) = 0, \forall x, y, z \in X.$$

Thus, by (25) we have for all $x, y, z \in X$

$$H\left(\varphi(x, y, z)\right) = \lim_{k \to \infty} \frac{1}{2^k} f\left(\varphi(2^k x, y, z)\right) = \lim_{k \to \infty} \frac{1}{2^k} \left(\psi\left(f\left(\frac{2^k x}{2^k}, f(y), f(z)\right) + Q\left(2^k x, y, z\right)\right)\right) = \psi\left(H(x), f(y), f(z)\right).$$

From the last equation and the additivity of $H$, we obtain that

$$H\left(\varphi(x, y, z)\right) = \frac{1}{2^k} H\left(\varphi(x, y, 2^k z)\right) = \psi\left(H(x), f(y), \frac{1}{2^k} f(2^k z)\right).$$

Letting $k \to \infty$ yields (21). This completes the proof. \(\square\)
From proving the theorems we have corollarys:

**Corollary 4.1.** Let $\mathcal{R}$ be a ring with a unit $1$ and $\mathcal{Y}$ be Banach algebra with a unit $e$ and $\epsilon, \delta \geq 0$ and $n \in \mathbb{N}, n \geq 2$. If a mapping $f : \mathcal{R} \rightarrow \mathcal{Y}$ satisfies

$$\left\| f \left( \sum_{j=1}^{n} x_{j} + \frac{1}{n} \sum_{j=1}^{n} x_{n+j} \right) - \sum_{j=1}^{n} f(x_{j}) - \frac{1}{n} \sum_{j=1}^{n} f \left( \frac{x_{n+j}}{n} \right) \right\|_Y \leq \epsilon \quad (26)$$

and

$$\left\| f \left( \prod_{j=1}^{n} x_{j} \prod_{j=1}^{n} x_{n+j} \right) - \prod_{j=1}^{n} f(x_{j}) - \frac{1}{n} \prod_{j=1}^{n} f \left( \frac{x_{n+j}}{n} \right) \right\|_Y \leq \delta \quad (27)$$

for all $x_1, x_2, \ldots, x_{2n} \in \mathcal{R}$ and $f(1) = e$, then there exists a unique ring homomorphism $H : \mathcal{R} \rightarrow \mathcal{Y}$ such that

$$\left\| f(x) - H(x) \right\|_Y \leq \frac{1}{2n-1} \epsilon, \forall x \in \mathcal{R}. \quad (28)$$

**Corollary 4.2.** Let $\mathcal{X}$ be an algebra, $\mathcal{Y}$ be Banach algebra and $\epsilon, \delta \geq 0$. If a mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ satisfies

$$\left\| f \left( \frac{x+y}{2} + z \right) - f \left( \frac{x+y}{2} \right) - f(z) \right\|_Y \leq \epsilon \quad (29)$$

and for all $x, y, z \in \mathcal{X}$,

$$\left\| f \left( [x,y,z] \right) - \left[ f(x), f(y), f(z) \right] \right\|_Y \left( \text{resp.} \left\| f \left( x \circ y \circ z \right) - f(x) \circ f(y) \circ f(z) \right\|_Y \right) \leq \delta, \quad (30)$$

then there exists a unique additive mapping $H : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$H \left( [x,y,z] \right) = \left[ H(x), H(y), H(z) \right], \left( \text{resp.} f \left( x \circ y \circ z \right) = f(x) \circ f(y) \circ f(z) \right), \quad (31)$$

for all $x, y, z \in \mathcal{X}$ and

$$\left\| f(x) - H(x) \right\|_Y \leq \epsilon, \forall x \in \mathcal{X}. \quad (32)$$

**References**


