Asia Mathematika
Volume: 4 Issue: 2 , (2020) Pages: 176 – 180
Available online at **www.asiamath.org**

Nondefinability in some expansions of the real field by power functions and the Borel mapping

Mourad Berraho[1*]
Department of Mathematics, Faculty of Sciences, Ibn Tofail University,
Kenitra, Morocco. Orchid iD: 0000-0003-2835-564X

Received: 13 Jun 2020 • **Accepted:** 26 Jul 2020 • **Published Online:** 24 Aug 2020

Abstract: In this paper, we firstly show that the germ of the sin function at the origin is not definable in the structure $(\mathbb{R},+,-,.,0,1,<,x^{\alpha_1},...,x^{\alpha_p})$, where the real numbers $\alpha_1,...,\alpha_p$ are $\mathbb{Q}$-linearly independent. Afterwards, we will investigate the Borel mapping over some quasianalytic rings.

Key words: Nondefinability, quasianalytic ring, the Borel mapping.

1. Introduction

It is well known thanks to [6] that the o-minimal structure $\mathcal{M} := (\mathbb{R},+,-,.,0,1,<,x^{\alpha_1},...,x^{\alpha_p})$ where $\alpha_1,...,\alpha_p$ are a real irrational numbers is polynomially bounded and model complete, so, it is interesting as it is polynomially bounded and generated by real analytic functions which are not globally subanalytic.
Firstly, we are going to show by following the proof given in [3] that the germ of the sin function at zero is not definable in the structure $(\mathbb{R},+,-,.,0,1,<,x^{\alpha_1},...,x^{\alpha_p})$, where $\alpha_1,...,\alpha_p$ are $\mathbb{Q}$-linearly independent real numbers.
Let $\mathcal{E}_1$ denote the ring of germs at the origin in $\mathbb{R}$ of $\mathcal{C}^\infty$ functions in a neighborhood of $0 \in \mathbb{R}$ and $\mathbb{R}[[x_1]]$ the ring of formal series with real coefficients. If $f \in \mathcal{E}_1$, we denote by $\hat{f} \in \mathbb{R}[[x_1]]$ its (infinite) Taylor expansion at the origin. The mapping $\mathcal{E}_1 \ni f \to \hat{f} \in \mathbb{R}[[x_1]]$ is called the Borel mapping. In other words, the Borel mapping takes germs at the origin in $\mathbb{R}$ of smooth functions to the sequence of the iterated partial derivatives at 0. A subring $\mathcal{C}_1 \subseteq \mathcal{E}_1$ is called quasianalytic if the restriction of the Borel mapping to $\mathcal{C}_1$ is injective.
It is a classical result due to Carleman [4] that the Borel mapping restricted to the germs at 0 of functions in a quasianalytic Denjoy-Carleman classes is never onto and the proof given in [8] is direct just by using techniques from Hilbert space. But this problem is difficult to study over an arbitrary quasianalytic subring of the ring of smooth germs $\mathcal{E}_1$ because we have no control over the growth of the derivatives of the functions belonging to such rings. Finally, we examine this mapping over some quasianalytic subrings of the ring of smooth germs.

2. Nondefinability of the sin germ at the origin of $\mathbb{R}$ in the structure $\mathcal{M}$

Let $\overline{\mathbb{R}} := (\mathbb{R},+,-,.,0,1,<)$ be the ordered field of real numbers. The main aim of this section is to show by using techniques from [3] and by following the proof given in ([2], Section 4) that the sin germ at 0 is not

*Correspondence: b.mourad87@hotmail.com

definable in the structure $(\overline{\mathbb{R}}, x^{\alpha_1}, ..., x^{\alpha_p})$ for all $\mathbb{Q}$-linearly independent real numbers $\alpha_1, ..., \alpha_p$.

Theorem 2.1. *Let $\alpha_1, ..., \alpha_p$ be an irrational real numbers, then the germ at $0 \in \mathbb{R}$ of the* sin *function is not definable in the structure $\mathcal{M} = (\overline{\mathbb{R}}, x^{\alpha_1}, ..., x^{\alpha_p})$.*

Proof. Let u denote the germ at 0 of the function $x \to \sin(x+1)$. It suffices to prove the theorem for $p = 1$, for this aim suppose that the sin germ at 1 is definable in the structure $(\overline{\mathbb{R}}, x^{\alpha_1})$, by [6, Section 3] the theory of the structure $(\overline{\mathbb{R}}, x^{\alpha_1})$ is model complete. So this germ is definable by an existential formula of the form $\exists y F(x, \sin(x), y) = 0$, where the term F is a polynomial in $x, \sin, y = (y_1, ..., y_n)$, and in $x^{\alpha_1}, y^{\alpha_1} := (y_1^{\alpha_1}, ..., y_n^{\alpha_1})$.

Therefore, in a vicinity of 0, the function $x \to \sin(x+1)$ is definable by the same formula $\exists y F(x, \sin(x+1), y) = 0$, where the term F is a polynomial in $x, u, y = (y_1, ..., y_n)$, and in $(x+1)^{\alpha_1}, (y+1)^{\alpha_1} := ((y_1+1)^{\alpha_1}, ..., (y_n+1)^{\alpha_1})$.

Desingularization. The restriction of the germ at 0 of the function u is definable by a formula of the form $\exists y \bigwedge_{i=1}^{n+1} h_i(x_0, x_1, y) = 0$,
$y = (y_1, ..., y_n)$ satisfying

$$\frac{\partial(h_1, ..., h_{n+1})}{\partial(x_1, y)}(x_0, x_1, y) \neq 0,$$

on the points (x_0, x_1, y) for which $h_i(x_0, x_1, y) = 0$, $i = 1, ..., n+1$.

Proof of the Desgulirization. See [3, Section 2.2, Case 2].

□

We can also assume that the terms h_j are "polynomials" on the variables shown or a constant times a variable and on the functions z^{α_1} where z is either one of the variables shown or a constant times a variable.

By the implicit function theorem, there exist a functions f_i defined in a neighborhood I of 0 such that $h_i(x, u(x), f_1(x), ..., f_n(x)) = 0$, $i = 1, .., n+1$, for all $x \in I$.

Also by applying translations to the variables $y_1, ..., y_n$ and changing accordingly the h_i, $i = 1, .., n+1$, we can assume that $f_i(x) > 0$ for all $x \in I$.

The fact that the functions $ix, ln(x+1), \alpha_1 ln(x+1); \ln(f_1+1), \alpha_1 \ln(f_1+1), ..., \ln(f_n+1), \alpha_1 \ln(f_n+1)$ are $\mathbb{Q}$-linearly independent is due to the irrationality of the number α_1 and by taking n minimal such that we have the above formula defining $u(x)$. If these functions were not linearly independent over $\mathbb{Q}$, by a linear change of variables we could decrease the number of variables needed to define $u(x)$, which contradicting the minimality of n.

Thanks to the formula $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$ for all real x,
we deduce that the transcendence degree of the ring $\mathbb{C}[ix, (ln(x+1), \alpha_1 ln(x+1)), (\ln(f_1+1), \alpha_1 \ln(f_1+1)), ..., (\ln(f_n+1), \alpha_1 \ln(f_n+1)), e^{ix}, x+1, (x+1)^{\alpha_1}, f_1+1, (f_1+1)^{\alpha_1}, ..., f_n+1, (f_n+1)^{\alpha_1}]$ over $\mathbb{C}$ is equal to the transcendence degree of the ring $\mathbb{C}[x, (ln(x+1), \alpha_1 ln(x+1)), (\ln(f_1+1), \alpha_1 \ln(f_1+1)), ..., (\ln(f_n+1), \alpha_1 \ln(f_n+$

$1)), \sin(x), (x+1)^{\alpha_1}, f_1, (f_1+1)^{\alpha_1}, ..., f_n, (f_n+1)^{\alpha_1}]$ over $\mathbb{C}$.

By Ax's Theorem in [7], the transcendence degree of the ring $\mathbb{C}[x, (ln(x+1), \alpha_1 ln(x+1)), (\ln(f_1+1), \alpha_1 \ln(f_1+1)), ..., (\ln(f_n+1), \alpha_1 \ln(f_n+1)), \sin(x), (x+1)^{\alpha_1}, f_1, (f_1+1)^{\alpha_1}, ..., f_n, (f_n+1)^{\alpha_1}]$ over $\mathbb{C}$ is at least $2n+4$.

So the transcendence degree over $\mathbb{C}$ of the ring $\mathbb{C}[x, f_1, ..., f_n, \sin(x), (x+1)^{\alpha_1}, (f_1+1)^{\alpha_1}, ..., (f_n+1)^{\alpha_1}]$ is at least $(2n+4)-(n+1) = n+3$.

By ([5], Theorem 26.5 (p. 202) and Theorem 30.3 ($1 \Leftrightarrow 4$)), if we have a functions $f_1, ..., f_n$ and polynomials $P_1(X_1, ..., X_n), ..., P_k(X_1, ..., X_n)$
(say, with $k < n$), whose Jacobian matrix has maximum rank at the point $(f_1, ..., f_n)$, then the transcendence degree of $\mathbb{C}[f_1, ..., f_n]/I$ over $\mathbb{C}$, where I is the ideal generated by $P_1(f_1, ..., f_n), ..., P_k(f_1, ..., f_n)$ is at most $n-k$.

So in our case, we put for all $i = 1, ..., n+1$,
$P_i(x, \sin(x), y_1, ..., y_n, (x+1)^{\alpha_1}, (y_1+1)^{\alpha_1}, ..., (y_n+1)^{\alpha_1}) = h_i(x, \sin(x), y_1, ..., y_n)$. As the ideal I is null, we deduce that the transcendence degree of

$$\mathbb{C}[x, \sin(x), f_1, ..., f_n, (x+1)^{\alpha_1}, (f_1+1)^{\alpha_1}, ..., (f_n+1)^{\alpha_1}]$$

over $\mathbb{C}$ is at most $(2n+3)-(n+1) = n+2$.
Which give us the desired contradiction. So, we conclude that the sin germ at 1 is not definable in the structure $\mathcal{M}$.

Suppose that the germ at 0 of the sin function is definable in the structure $\mathcal{M}$, then so does the germ at 1 of the sin function thanks to the formula $\sin(x+1) = \sin(x)\cos(1) + \sin(1)\cos(x)$.

□

By following the proof of Theorem 2.1 and for a $\mathbb{Q}$-linearly independent real numbers $\alpha_1, ..., \alpha_p$, it is not hard to see that by deleting the functions $\alpha_1 \ln(f_1+1), ..., \alpha_p \ln(f_1+1)$, the transcendence degree decreases just by 1, therefore, we deduce the following theorem to end this section.

Theorem 2.2. *The germ of the* sin *function at the origin of* $\mathbb{R}$ *is not definable in the structure* $(\overline{\mathbb{R}}, x^{\alpha_1}, ..., x^{\alpha_p})$ *for all* $\mathbb{Q}$*-linearly independent real numbers* $\alpha_1, ..., \alpha_p$.

3. Remarks on the Borel mapping over some quasianalytic rings

In this section, we are going to study the Borel mapping over some quasianalytic subrings of the ring of smooth germs $\mathcal{E}_1$ in light of the results described in ([1], Section 3).

Definition 3.1. Let $\mathcal{C}_1 \subset \mathcal{E}_1$ be a subring of the ring of germs of smooth functions at the origin of $\mathbb{R}$. We say that $\mathcal{C}_1$ is a quasianalytic ring if the Borel mapping $\hat{} : \mathcal{C}_1 \to \mathbb{R}[[x_1]]$ is injective.

Example 3.1. *The ring of real analytic germs is a quasianalytic ring.*

By quasianaliticity, we may assume that $\hat{\mathcal{C}_1} \subset \mathbb{R}[[x_1]]$.

Assume that these quasianalytic rings satisfy the following property called the stability under monomial division:

Let $\hat{f} \in \hat{\mathcal{C}_1}$ and $\hat{f} = x_1\widehat{\varphi}$ where $\widehat{\varphi} \in \mathbb{R}[[x_1]]$, then $\varphi \in \mathcal{C}_1$.

Remark 3.1. *The ring $\mathcal{C}_1$ is a principal ideal domain, (see Remark 3.1 in [1]).*

Proposition 3.1. *Suppose that the Borel mapping $^\wedge : \mathcal{C}_1 \to \mathbb{R}[[x_1]]$ is surjective, then $\mathcal{C}_1$ is algebraically closed in $\mathcal{E}_1$.*

Proof. See [1].

□

Remark 3.2. *The reciprocal of proposition 3.1 does not hold. Indeed, let's take the ring $\mathcal{A}_1$ to be the ring of germs at $0 \in \mathbb{R}$ of real analytic functions. So, this ring is quasianalytic and closed under monomial division. By [9, Theorem 1], the ring $\mathcal{A}_1$ is algebraically closed in $\mathcal{E}_1$. But the Borel mapping $^\wedge : \mathcal{A}_1 \to \mathbb{R}[[x_1]]$ is not surjective.*

□

We put,

$$\mathcal{F} = \{\mathcal{B} \subset \mathcal{E}_1/\mathcal{B} \text{ is a quasianalytic ring, closed under derivation } \}.$$

Proposition 3.2. *If the ring $\mathcal{C}_1$ is a maximal element of $\mathcal{F}$, then the (x_1)-adic completion of the ring $\hat{\mathcal{C}_1}$ is the ring of formal power series $\mathbb{R}[[x_1]]$.*

Proof. If the ring $\mathcal{C}_1$ is a quasianalytic and maximal element of $\mathcal{F}$, and P a polynomial of degree n, then the ring $\mathcal{C}_1(P, P', ..., P^{(n)})$ is also closed under derivation and a quasianalytic ring, as the ring $\mathcal{C}_1$ is a maximal element of $\mathcal{F}$, the ring $\mathcal{C}_1$ contains the polynomial ring $\mathbb{R}[x_1]$, we deduce that the (x_1)-adic completion of the ring $\hat{\mathcal{C}_1}$ is the ring of formal power series $\mathbb{R}[[x_1]]$.

□

Proposition 3.3. *Let $\mathcal{C}_1 \subset \mathcal{E}_1$ be quasianalytic ring such that there exist a quasianalytic ring $\mathcal{D}_1$ that contains strictly the ring $\mathcal{C}_1$, then the Borel mapping $^\wedge : \mathcal{C}_1 \to \mathbb{R}[[x_1]]$ is not surjective.*

Proof. Suppose otherwise, let $f \in \mathcal{D}_1 \backslash \mathcal{C}_1$. Since the Borel mapping $^\wedge$ is surjective, there exists $g \in \mathcal{C}_1$ such that $\hat{f} = \hat{g}$. We get $f = g \in \mathcal{C}_1$ by the quasianalycity of $\mathcal{D}_1$. Contradiction.

□

Problem : Let $\mathcal{C}_1$ be a maximal element of $\mathcal{F}$, it is not clear to us the answer to the question of the surjectivity of the Borel mapping $^\wedge : \mathcal{C}_1 \to \mathbb{R}[[x_1]]$ in case there is no other quasianalytic ring containing the ring $\mathcal{C}_1$.

References

[1] M. Berraho, *On a problem concerning the ring of Nash germs and the Borel mapping*, Aims Mathematics, 5(2): 923–929. January 2020.

[2] M. Berraho, *On definable germs of functions in expansions of the real field by power functions*, Asia Mathematika, 1-10. August 2020.

[3] R. Bianconi, *Nondefinability results for expansions of the field of real numbers by the exponential function and by the restricted sine function*, The Journal of Symbolic Logic 62 (4) (1997), 1173-1178.

[4] T. Carleman, *les fonctions quasi-analytiques, Gauthiers Villars Paris (1926)*,

[5] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, 1986.

[6] C. Miller, *Expansions of the real field with power functions*, Ann. Pure Appl. Logic 68(1994), no. 1, 79-94.

[7] T. Servi, *On local interdefinability of (real and complex) analytic functions*, Université Paris Diderot, 1st April 2018.

[8] V. Thielliez, *On quasianalytic local rings*, Expo. Math., 26 (1): 1-23, 2008.

[9] V. Thielliez, *Smooth solutions of quasianalytic or ultraholomorphic equations*, July 2010, Volume 160, Issue 4, pp 443-453.