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On definable germs of functions in expansions of the real field by power functions

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Abstract: We recall the notion of $\mathbb{R}$-analytic functions, these are definable in an o-minimal expansion of the real field $\mathbb{R}$ and are locally the restriction of a holomorphic definable function. In this paper, we prove that if the Weierstrass division theorem holds for the ring of smooth definable germs, then every definable real analytic germ is $\mathbb{R}$-analytic and that the converse does not hold. In the same section, for the structure $\mathcal{M} = (\mathbb{R}, +, -, ., 0, 1, <, x^{\alpha_1}, ..., x^{\alpha_p})$ where $\alpha_1, ..., \alpha_p$ are an irrational numbers, we show that the ring of $\mathbb{R}$-analytic germs is equal to the ring of Nash germs. Finally, we show that the germ of the exp function at zero is definable neither in the structure $\mathcal{M}$ nor in the structure $(\mathbb{R}, +, -, ., 0, 1, <, x^{\alpha_1}, ..., x^{\alpha_p})$, where the real numbers $\alpha_1, ..., \alpha_p$ are $\mathbb{Q}$-linearly independent.

Key words: Weierstrass division theorem, polynomially bounded o-minimal structures, Nash germs.

1. Introduction

In this paper, we are going to study some properties of the o-minimal structure $\mathcal{M} := (\mathbb{R}, +, -, ., 0, 1, <, x^{\alpha_1}, ..., x^{\alpha_p})$ where $\alpha_1, ..., \alpha_p$ are a real irrational numbers. By [2] this structure is polynomially bounded and model complete so it is interesting as it is polynomially bounded and generated by real analytic functions which are not globally subanalytic. Ricardo Bianconi proved in [13] that if $\beta \in \mathbb{R}$ such that β is not in the field generated by $\alpha_1, ..., \alpha_p$, then no restriction of the function x^β to an interval is definable in the o-minimal structure $\mathcal{M}$.

A little more generally, the interesting o-minimal structures that are expansions of an arbitrary real closed fields R by adding new functions or relations have been important tools in proving theorems in other areas as algebraic geometry for example thanks to the quantifier elimination of the structure $(R, +, -, ., <, 0, 1)$, the semialgebraic sets are closed under projection (see [4], Corollary 3.3.18). The reader should consult [10] for an account of o-minimal

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structures of a real closed field.

Given an open set $U \subset \mathbb{R}^n$, a definable function $f : U \to \mathbb{R}$ in an o-minimal expansion of the real field is called an $\mathbb{R}$-analytic function if for every $a \in U$, there is an holomorphic definable function on an open neighbourhood of a such that its restriction to $\mathbb{R}^n$ coincides with f around a. It is easy to check that every $\mathbb{R}$-analytic function is a real analytic and definable function, contrary to the complex case, not every real analytic and definable function is $\mathbb{R}$-analytic as it is shown in ([15], Example 2.1) for the structure $(\mathbb{R}, +, -, ., 0, 1, <, \exp)$.

The Weierstrass division theorem is the key tool for local complex analytic geometry (see for example Gunning and Rossi [14, Chapter II]). It is also used for example in the proof of the important Oka's coherence theorem (see [14, Chapter IV]). In this connexion, the following question asked by Lou Van Den dries in [9]: Does the Weierstrass division theorem hold for the ring of germs of real analytic definable functions in an o-minimal structure (not necessarily polynomially bounded) extending the structure of the real numbers? in [15], there is a positive answer in the semialgebraic setting and also in the structure of globally subanalytic sets and functions but a negative answer in the structure $(\mathbb{R}, +, -, ., 0, 1, <, \exp)$, so for the structure $\mathcal{M}$, we deduce a positive answer for its subring of $\mathbb{R}$-analytic functions. But for an o-minimal structure expanding the real field, in [16], it is shown a piecewise Weierstrass division theorem for the definable holomorphic functions. Also in [1], it is shown in particular that the Weierstrass division theorem does not hold for the ring of the smooth germs that are definable in a polynomially bounded o-minimal structure that contains strictly the ring of real analytic germs.

We firstly show that for the structure $\mathcal{M}$, the ring of $\mathbb{R}$-analytic germs is equal to the ring of Nash germs.

Nondefinability results are a stronger forms of independence ones, so in the last section, we are going to show in the light of [12] that the germ of the exp function at zero is not definable in the structure $\mathcal{M}$. Actually, we can prove that the germ of the exp function at zero is not definable in the structure $(\mathbb{R}, +, -, ., 0, 1, <, x^{\alpha_1}, ..., x^{\alpha_p})$, where $\alpha_1, ..., \alpha_p$ are $\mathbb{Q}$-linearly independent real numbers.

2. Preliminaries.

Throughout this paper, we will work over some fixed o-minimal expansions $\mathcal{R}$ of the structure $\overline{\mathbb{R}} := (\mathbb{R}, +, -, ., 0, 1, <)$ in a first order language extending $\{+, -, ., 0, 1, <\}$. Definable means definable with parameters from $\mathbb{R}$. A function $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ is said to be definable if its graph is definable. We say that the structure $\mathcal{R}$ is o-minimal if the definable subsets of $\mathbb{R}$ are just finite unions of intervals of all kinds, including singletons. This structure is polynomially bounded if for every definable function $f : \mathbb{R} \to \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $|f(t)| \leq t^N$

for all sufficiently large positive t.

Example 2.1

- The structure $\overline{\mathbb{R}} := (\mathbb{R}, +, -, ., 0, 1, <)$ is polynomially bounded and o-minimal (by Tarski-Seidenberg); the sets definable in this structure are exactly the semialgebraic sets.
- The structure of the ordered real field with restricted analytic functions $\mathbb{R}_{an}$ whose definable sets are the finitely subanalytic sets is a polynomially bounded o-minimal structure by [2].

Denote by $\mathcal{E}_n$ the ring of smooth function germs at the origin of $\mathbb{R}^n$, and by $\mathcal{C}_n$ a sequence of its local subrings, we put $x = (x_1, ..., x_n)$.

For every $f \in \mathcal{E}_n$, we denote by $\hat{f} \in \mathbb{R}[[x_1, ..., x_n]]$ its infinite Taylor expansion at the origin of $\mathbb{R}^n$.

The mapping $\mathcal{E}_n \ni f \mapsto \hat{f} \in \mathbb{R}[[x_1, ..., x_n]]$ is called the Borel mapping.

We say that the local rings $\mathcal{C}_n$ are quasianalytic if the Borel mapping $\wedge : \mathcal{C}_n \to \mathbb{R}[[x]]$ is injective for each $n \in \mathbb{N}$. For example, by [3] the set of smooth function germs that are definable in a polynomially bounded o-minimal structure is a system of quasianalytic local rings whose maximal ideal are generated by the germ at zero of the coordinate functions $x \to x_i : \mathbb{R}^n \to \mathbb{R}$, for each $n \in \mathbb{N}$.

In the sequel of this section, we will recall another interesting and famous example of a quasianalytic local rings:

We use the following notation : for any multi-index $J = (j_1..., j_n)$ of $\mathbb{N}^n$, we denote the length $j_1 + ... + j_n$ of J by the corresponding lower case letter j. We put $D^J = \partial^j / \partial x_1^{j_1} ... \partial x_n^{j_n}$, $J! = j_1!...j_n!$ and $x^J = x_1^{j_1} ... x_n^{j_n}$, where $x = (x_1, ..., x_n)$.

Let $M = (M_j)_j$ be an increasing sequence of positive real numbers, with $M_0 = 1$. We define the Denjoy-Carleman class $\mathcal{E}_n(M)$ to be the set of smooth germs f for which there exist a neighborhood U of 0 and a positive constants C and σ such that

$$|D^J f(x)| \leq C\sigma^j j! M_j \text{ for any } J \in \mathbb{N}^n \text{ and } x \in U.$$

Here, $C\sigma^j j!$ appears as "the analytic part" of the estimate, whereas M_j can be considered as a way to allow a defect of analyticity, if $\mathcal{O}_n$ denotes the ring of the real analytic functions germs at the origin of $\mathbb{R}^n$, we clearly have

$$\mathcal{O}_n \subset \mathcal{E}_n(M) \subset \mathcal{E}_n.$$

Frow now on, we shall always make the following assumption:

the sequence M is logarithmically convex .

This amounts to saying that the sequence M_{j+1}/M_j increases.

This assumption implies that the class $\mathcal{E}_n(M)$ is a local ring with maximal ideal $\{h \in \mathcal{E}_n(M) : h(0) = 0\}$.

- The local ring $\mathcal{E}_n(M)$ is quasianalytic if and only if

$$\sum_{j=0}^{+\infty} \frac{M_j}{(j+1)M_{j+1}} = \infty.$$

- We have $\mathcal{O}_n = \mathcal{E}_n(M)$ if and only if $\sup_{j\geq 1}(M_j)^{1/j} < \infty$.

- The ring $\mathcal{E}_n(M)$ is stable under derivation if and only if

$$\sup_{j\geq 1}(M_{j+1}/M_j)^{1/j} < \infty.$$

The interested reader should see [18] for a thorough treatment of the rings $\mathcal{E}_n(M)$.

3. Weierstrass division theorem over the structure $\mathcal{M}$

We begin this paper by this fundamental remark that gives an easier proof of a Bianconi's result (see[12]) thanks to the following theorem proved in [13].

Theorem 3.1. *(see[13]). Let $D \subset \mathbb{R}^{2n}$ be a definable open polydisc and $u, v : D \to \mathbb{R}$ two definable functions in $(\overline{\mathbb{R}}, \exp)$ (with parameters from $\mathbb{R}$) such that $f(x+iy) = (u+iv)(x,y)$ is holomorphic in D. Then u and v are already definable in $\overline{\mathbb{R}}$.*

Remark 3.1. *In particular if $n=1$ and $D=\mathbb{R}^2$, by [11, proposition 4.1] the functions u and v are polynomials, so the theorem is trivial in this case.*

Remark 3.2. *We recall Bianconi's Theorem proved in [12] that states that no restriction of the sine function to an interval is definable in the structure $(\overline{\mathbb{R}}, \exp)$. Indeed, suppose that for some $a>0$, $\sin|_{[-a,a]}$ is definable in $(\overline{\mathbb{R}}, \exp)$, we know that $\exp(x+iy)$ is holomorphic and that its real part is $u(x,y) := e^x\cos(y)$ and its imaginary part is $v(x,y) := e^x\sin(y)$, so the functions u and v are definable in the structure $(\overline{\mathbb{R}}, \exp)$ on the set $([-a,a])^2$, so by Theorem 2.1, we deduce that these functions u and v are also definable in the structure $\overline{\mathbb{R}}$, so $e^{2x}(\cos^2(y)+\sin^2(y)) = e^{2x}$ is also definable in $\overline{\mathbb{R}}$, but $x \to e^x$ is not a semialgebraic function, which is a contradiction.*

3.1. Weierstrass division theorem over an arbitrary polynomially bounded o-minimal structure

Fix a polynomially bounded o-minimal structure $\mathcal{R}$ that is an expansion of the structure $\overline{\mathbb{R}} = (\mathbb{R}, <, 0, 1, +, -, .)$ and denote by $\mathcal{R}_n^\infty$ the ring of those smooth functions germs at $0 \in \mathbb{R}^n$ which are definable in $\mathcal{R}$ and by $\mathcal{R}_n^\omega$ the ring of real analytic functions germs at 0 that are definable in $\mathcal{R}$.

We will not distinguish notationally between a function and its germ.

We know by [3] that each ring $\mathcal{R}_n^\infty$ is a quasianalytic local ring, so we may suppose that $\mathcal{R}_n^\infty \subset \mathbb{R}[[x]]$, (where $x=(x_1,...,x_n)$).

Let $(\mathcal{D}_n^{\mathcal{R}})_{n\in\mathbb{N}}$ denote the system of the rings of $\mathbb{R}$-analytic germs at $0 \in \mathbb{R}^n$ (see the definition above in the introduction).

We firstly show that if the system of the rings $(\mathcal{R}_n^\infty)_{n\in\mathbb{N}}$ satisfies Weierstrass division theorem, then $\mathcal{R}_n^\omega = \mathcal{D}_n^{\mathcal{R}}$ for all $n \in \mathbb{N}$.

Proposition 3.1. *Suppose that the system of the rings $(\mathcal{R}_n^\infty)_{n\in\mathbb{N}}$ satisfies Weierstrass division theorem, then the system of the rings $(\mathcal{R}_n^\omega)_{n\in\mathbb{N}}$ is equal to the system of the $\mathbb{R}$-analytic germs $(\mathcal{D}_n^{\mathcal{R}})_{n\in\mathbb{N}}$.*

Proof. Let $f, g \in \mathcal{R}_n^\omega$ such that g is regular of order p with respect to X_n, by the Weierstrass division theorem in the ring $\mathcal{R}_n^\infty$, we deduce that there exist unique $\tilde{q} \in \mathcal{R}_n^\infty$,

$\tilde{r}_1, ..., \tilde{r}_p \in \mathcal{R}^{\infty}_{n-1}$ such that

$$\hat{f} = \hat{g}\tilde{q} + \sum_{j=1}^{p} \tilde{r}_j(X_1, ..., X_{n-1})X_n^{p-j}.$$

It is well known by ([5], Theorem 6.7) that the ring of real analytic germ satisfies Weierstrass division theorem, so there exist unique $q \in \mathcal{O}_n$ and $r_1, ..., r_p \in \mathcal{O}_{n-1}$ such that

$$\hat{f} = \hat{g}q + \sum_{j=1}^{p} r_j(X_1, ..., X_{n-1})X_n^{p-j}.$$

As $\mathcal{R}^{\omega}_n \subset \mathcal{R}^{\infty}_n$ and thanks to the unicity of the division in the ring $\mathcal{R}^{\infty}_n$, the functions q and $r_1, ..., r_p$ are also definable in the structure $\mathcal{R}$, we deduce that the system of the rings $(\mathcal{R}^{\omega}_n)_{n\in\mathbb{N}}$ also satisfies the Weierstrass division theorem. By ([15, Theorem 2.4]), we conclude that $\mathcal{R}^{\omega}_n = \mathcal{D}^{\mathcal{R}}_n$ for all $n \in \mathbb{N}$.

□

Remark 3.3. *The reciprocal of Proposition 3.1 is not true.*

Proof of Remark 3.3. Let $\mathcal{E}_n(M)$ denote a non-analytic and a quasianalytic Denjoy-carleman ring (see the previous section). For each $n \in \mathbb{N}^*$, let $f \in \mathcal{E}_n(M)$, we define $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ by $\widetilde{f}(x) = f(x)$ if $x \in [-1, 1]^n$ and $\widetilde{f}(x) = 0$ otherwise. We let $\mathbb{R}_{\mathcal{E}_n(M)} := (\overline{\mathbb{R}}, (\widetilde{f})_{f\in\mathcal{E}_n(M)})$ be the expansion of the real field by all restricted functions $\widetilde{f}$ for $f \in \mathcal{E}_n(M)$, let's consider the polynomially bounded o-minimal structure $\mathcal{R} := \{\mathbb{R}_{\mathcal{E}_n(M)}; n \in \mathbb{N}\}$ which is the expansion of the real field by the restricted functions in $\mathcal{E}_n(M)$ (see [8]), so the ring of real analytic germs $\mathcal{O}_n$ is contained in the ring of smooth germs that are definable in the structure $\mathbb{R}_{\mathcal{E}_n(M)}$ which we denote by $\mathcal{R}^{\infty}_n$. As the ring of real analytic germs $\mathcal{O}_n$ is contained in the Denjoy-Carleman ring $\mathcal{E}_n(M)$, we deduce that the ring of real analytic definable germs in the structure $\mathcal{R}$ (denoted $\mathcal{R}^{\omega}_n$) is equal to the ring of real analytic germs, so the Weierstrass division theorem holds for the system of the rings $\mathcal{R}^{\omega}_n$. By ([15, Theorem 2.4]), the system $\mathcal{R}^{\omega}_n$ is exactly equal to the system $(\mathcal{D}^{\mathcal{R}}_n)_{n\in\mathbb{N}}$.

As the system of the rings $\mathcal{R}^{\infty}_n$ satisfies all the requirements of Definition 2.2 in [1], so if these rings satisfy Weierstrass division theorem, then by ([5], Lemma 6.5), these rings also satisfy Weierstrass preparation theorem, by the main result of [1], we deduce that these rings $\mathcal{R}^{\infty}_n$ are equal to the ring of real analytic germs $\mathcal{O}_n$ for all $n \in \mathbb{N}$, which is a contradiction.

□

Remark 3.4. *Let's take the structure* $\mathcal{R} = (\overline{\mathbb{R}}, \exp)$*, so by Example 2.1 in [15], the system* $(\mathcal{R}_n^\omega)_{n\in\mathbb{N}}$ *does not satisfy Weierstrass division theorem, so thanks to Proposition 3.1, we deduce easily that the system* $(\mathcal{R}_n^\infty)_{n\in\mathbb{N}}$ *does not satisfy Weierstrass division theorem.*

3.2. The structure $\mathcal{M}$

In the following we will confine ourselves just to the case when the structure $\mathcal{R}$ is equal to the structure $\mathcal{M} := (\overline{\mathbb{R}}, x^{\alpha_1}, ..., x^{\alpha_p})$ where $\alpha_1, ..., \alpha_p$ are arbitrary irrational numbers.

In this paper, we let $\mathcal{M}$ denote the o-minimal structure $(\overline{\mathbb{R}}, x^{\alpha_1}, ..., x^{\alpha_p})$ where $\alpha_1, ..., \alpha_p$ is a finite sequence of irrational numbers and by x^{α_i} the function which is equal to the usual x^{α_i} if $x > 0$ and equals to 0, if $x \leq 0$ for all $i = 1, ..., p$.

We recall that a Nash germ is a real analytic algebraic function germ. See ([7]) for a thorough treatment of semialgebraic functions and of Nash germs.

Let $\mathcal{N}_n$ denote the ring of Nash germs.

Theorem 3.2. $\mathcal{D}_n^{\mathcal{M}} = \mathcal{N}_n$.

Proof. Let $f \in \mathcal{D}_n^{\mathcal{M}}$, so by definition the germ f has a definable holomorphic extension F, let P and Q denote the real and the imaginary part of the function F, by ([15], Theorem 1.7) we deduce that P and Q are also $\mathbb{R}$-analytic germs, so they are also in the ring $\mathcal{D}_n^{\mathcal{M}}$, so thanks to [13, Corollary 1] these functions P and Q are a semialgebraic ones, so by viewing F as a function in $2n$ reals variables, F is also a semialgebraic germ, as f is a real analytic germ, we have that $f(x) = \sum_{k\in\mathbb{N}^n} a_k x^k$, so $f(x) = F(x, 0)$, therefore f is also a semialgebraic germ, as f is analytic, it is also a Nash germ. So $\mathcal{D}_n^{\mathcal{M}} \subseteq \mathcal{N}_n$.
Conversely, let $f \in \mathcal{N}_n$, so f is definable in $\mathcal{M}$ (as f is definable in $\overline{\mathbb{R}}$), so by applying ([15], Corollary 4.5), its holomorphic extension F is definable in $\overline{\mathbb{R}}$, so F is also definable in $\mathcal{M}$ and as f is real analytic, we deduce that $f \in \mathcal{D}_n^{\mathcal{M}}$.

□

4. Nondefinability of the exp germ at zero in the structure $\mathcal{M}$

The main aim of this section is to show by using techniques from [12] that the exp germ at 0 is not definable in the structure $(\overline{\mathbb{R}}, x^{\alpha_1}, ..., x^{\alpha_p})$ for all $\mathbb{Q}$-linearly independent real numbers $\alpha_1, ..., \alpha_p$.

Theorem 4.1. *Let $\alpha_1, ..., \alpha_p$ be an irrational real numbers, then the germ of the exponential function at the origin of $\mathbb{R}$ is not definable in the structure $\mathcal{M} = (\overline{\mathbb{R}}, x^{\alpha_1}, ..., x^{\alpha_p})$.*

Proof. It suffices to prove the theorem for $p = 1$, for this aim suppose that the exp germ at 0 is definable in the structure $(\overline{\mathbb{R}}, x^{\alpha_1})$, by [2, Section 3] the theory of the structure $(\overline{\mathbb{R}}, x^{\alpha_1})$ is model complete. So this germ is definable by an existential formula of the form $\exists y F(x, e^x, y) = 0$, where the term F is a polynomial in $x, e^x, y = (y_1, ..., y_n)$, and in $x^{\alpha_1}, y^{\alpha_1} := (y_1^{\alpha_1}, ..., y_n^{\alpha_1})$.

Therefore, the germ at 1 of the exp function is definable by the same formula $\exists y F(x, e^x, y) = 0$, where the term F is a polynomial in $x, e^x, y = (y_1, ..., y_n)$, and in $(x+1)^{\alpha_1}, (y+1)^{\alpha_1} := ((y_1+1)^{\alpha_1}, ..., (y_n+1)^{\alpha_1})$.
As $e^{x+1} = e^x e$ for all x in a vicinity of 0, we deduce that the germ of the exp germ at 0 is definable by this last formula up to a constant.

Let u denote the germ at 0 of the exp function.

Desingularization. The restriction of the germ at 0 of exp is definable by a formula of the form $\exists y \bigwedge_{i=1}^{n+1} h_i(x_0, x_1, y) = 0$, $y = (y_1, ..., y_n)$ satisfying

$$\frac{\partial(h_1, ..., h_{n+1})}{\partial(x_1, y)}(x_0, x_1, y) \neq 0,$$

on the points (x_0, x_1, y) for which $h_i(x_0, x_1, y) = 0$, $i = 1, ..., n+1$.

Proof of Desingularization. See [12, Section 2.2, Case 2].

□

We can also assume that the terms h_j are "polynomials" on the variables shown or a constant times a variable and on the functions z^{α_1} where z is either one of the variables shown or a constant times a variable.

By the implicit function theorem, there exist a functions f_i defined in a neighborhood I of 0 such that $h_i(x, u(x), f_1(x), ..., f_n(x)) = 0$, $i = 1, .., n+1$, for all $x \in I$.

Also by applying translations to the variables $y_1, ..., y_n$ and changing accordingly the h_i, $i = 1, .., n+1$, we can assume that $f_i(x) > 0$ for all $x \in I$.

The functions $x; \ln(x+1), \alpha_1 ln(x+1); \ln(f_1+1), \alpha_1 \ln(f_1+1); ...;$
$\ln(f_n+1), \alpha_1 \ln(f_n+1)$ are $\mathbb{Q}$-linearly independent comes from the fact that the number α_1 is irrational and by taking n minimal such that we have the above formula defining $u(x)$. If

these functions were not linearly independent over $\mathbb{Q}$, by a linear change of variables we could decrease the number of variables needed to define $u(x)$, which contradicting the minimality of n.

By Ax's Theorem in [17], the transcendence degree of $\mathbb{C}[x, (\ln(x+1), \alpha_1 \ln(x+1)), (\ln(f_1+1), \alpha_1 \ln(f_1+1)), ..., (\ln(f_n+1), \alpha_1 \ln(f_n+1)), e^x, (x+1)^{\alpha_1}, f_1, (f_1+1)^{\alpha_1}, ..., f_n, (f_n+1)^{\alpha_1}]$ over $\mathbb{C}$ is at least $2n+4$.

So the transcendence degree over $\mathbb{C}$ of the ring $\mathbb{C}[x, f_1, ..., f_n, e^x, (x+1)^{\alpha_1}, (f_1+1)^{\alpha_1}, ..., (f_n+1)^{\alpha_1}]$ is at least $(2n+4)-(n+1) = n+3$.

By ([6], Theorem 26.5 (p. 202) and Theorem 30.3 ($1 \Leftrightarrow 4$)) if we have a functions $f_1, ..., f_n$ and polynomials $P_1(X_1, ..., X_n), ..., P_k(X_1, ..., X_n)$ (say, with $k < n$), whose Jacobian matrix has maximum rank at the point $(f_1, ..., f_n)$, then the transcendence degree of $\mathbb{C}[f_1, ..., f_n]/I$ over $\mathbb{C}$, where I is the ideal generated by $P_1(f_1, ..., f_n), ..., P_k(f_1, ..., f_n)$ is at most $n-k$.

So in our case, we put for all $i = 1, ..., n+1$, $P_i(x, e^x, y_1, ..., y_n, (x+1)^{\alpha_1}, (y_1+1)^{\alpha_1}, ..., (y_n+1)^{\alpha_1}) = h_i(x, e^x, y_1, ..., y_n)$. As the ideal I is null, we deduce that the transcendence degree of

$$\mathbb{C}[x, e^x, f_1, ..., f_n, (x+1)^{\alpha_1}, (f_1+1)^{\alpha_1}, ..., (f_n+1)^{\alpha_1}]$$

over $\mathbb{C}$ is at most $(2n+3)-(n+1) = n+2$.
Which give us the desired contradiction. So we conclude that the exp germ at zero is not definable in the structure $\mathcal{M}$.

$\square$

By following the proof of theorem 4.1 and for a $\mathbb{Q}$-linearly independent real numbers $\alpha_1, ..., \alpha_p$, it is not hard to see that by deleting the functions $\alpha_1 \ln(f_1+1), ..., \alpha_p \ln(f_1+1)$, the transcendence degree decreases just by 1, therefore, we deduce the following theorem to end this paper.

Theorem 4.2. *The germ of the exponential function at the origin is not definable in the structure* $(\overline{\mathbb{R}}, x^{\alpha_1}, ..., x^{\alpha_p})$ *for all* $\mathbb{Q}$*-linearly independent real numbers* $\alpha_1, ..., \alpha_p$.

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