Commutativity theorems for rings with its applications

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Abstract: The present paper is to prove new commutativity theorems for rings (see Theorems 2.1, and 3.1). In addition, applications of commutativity theorems for rings to near-rings, we investigate some polynomial identities \((P_4)\) and \((P_5)\) with exponents depending on \(x\) and \(y\) for a certain class of near-rings and then, under some constraints, it is shown that D-near-ring as well as distributively generated \((d g)\) near-rings are commutative. Finally, we close our discussion with some open problems.

Key words: Commutativity, distributively generated \((d g)\) near-ring, D-near-ring, ring with identity, semiprime ring

1. Introduction

Throughout, \(R\) will be an associative ring (may be without unity 1), \(Z(R)\) will represent the centre of \(R\), \(N(R)\) the set of all nilpotent elements in \(R\), and \(C(R)\) the commutator ideal of \(R\). For any \(x, y \in R\), the commutator \(xy - yx\) will be denoted by \([x, y]\). By \(GF(q)\), we mean the Galois field (finite field) with \(q\) elements, and \((GF(q))_2\) the ring of all \(2 \times 2\) matrices over \(GF(q)\), and \(Z[X]\) is the totality of polynomials in \(X\) with coefficients in \(Z\), the ring of integers. A classical result in ring theory is the theorem of Herstein [13], which gives a condition on the commutators \(xy - yx\) of a ring that is necessary and sufficient for commutativity, and established the commutativity of \(R\) if, for each \(x, y \in R\) there exists a natural number \(n = n(x, y) > 1\) depending on \(x\) and \(y\), and \(R\) satisfies the condition \([x, y]^n = [x, y]\). Harmanci [12] proved the commutativity of a ring with unity 1 satisfying \([x^k, y] = [x, y^k], k = n, n+1\). Further, Gupta [11] extended it for the commutativity of a semiprime ring \(R\) with unity 1 when \(R\) satisfies \([x^m, y] - [x, y^m] \in Z(R)\) and \([x^{m+1}, y] - [x, y^{m+1}] \in Z(R)\), for all \(x, y \in R\). It is a natural question: “Whether the existence of unity 1 in the semiprime ring of Gupta’s [11] result is essential with both the identities are necessary?” Answering this question, for any \(x, y \in R\), and \(k\) and \(n\) are natural numbers, we investigate the ring properties:

\((P_1)\) There exists a positive integer \(m > 1\) such that either (i) \([x^n, y^m] + [x^m, y], x] = 0\) or (ii) \([x^n, y^m] - [x^m, y], x] = 0\).

\((P_2)\) There exists a positive integer \(m > 1\) such that either (i) \([x^n, y^m] + [x^m, y], y] = 0\) or (ii) \([x^n, y^m] - [x^m, y], y] = 0\).

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In addition, it is shown that the commutativity of semiprime rings without unity satisfying one of the conditions \((P_1)\) and \((P_2)\).

In Section 2, we first prove the conditions \((P_1)\) and \((P_2)\) for semi prime rings that is an extension of the results of [1] and [11], and provide one example to show that this result can not extend for arbitrary rings. Section 3 is devoted to commutativity of rings with unity 1 with later defined properties \((P_3)\) and \(Q(n)\), and it demonstrates an example that the hypotheses are not superfluous. Section 4 includes some applications of commutativity theorems in near-rings as; “contributions on commutativity of \(D\)-near-rings” as well as “distributively generated \((d\ g)\) near-rings”. Finally, we conclude it with some open problems in Section 5.

2. A commutativity theorem for semiprime rings

We begin with the following theorem.

**Theorem 2.1.** Let \(R\) be a semiprime ring. Then the following statements are equivalent.

a) \(R\) satisfies \((P_1)\).

b) \(R\) satisfies \((P_2)\).

c) \(R\) is a commutative ring.

Before proving Theorem 2.1, we need the following result.

**Lemma 2.1.** [6] Let \(R\) be a ring satisfying an identity \(q(X) = 0\), where \(q(X)\) is a polynomial in a finite number of noncommuting indeterminates, its coefficients being integers with highest common factor 1. If there exist no prime \(p\) for which the ring \(2 \times 2\) matrices over \(GF(p)\) satisfies \(q(X) = 0\), then \(R\) has a nil commutator ideal and the nilpotent elements of \(R\) form an ideal.

**Proof of Theorem 2.1**

Since a semiprime ring \(R\) is isomorphic to a sub direct sum of prime rings \(R_\alpha\), each of \(\alpha\), which as a homomorphic image of \(R\) satisfies the hypothesis placed on \(R\), we may assume that the ring \(R\) is prime satisfying one of the conditions (a) and (b). Given \(R\) satisfies (c), then (a) and (b) follows immediately.

Next, we claim that \((a) \implies (c)\) and \((b) \implies (c)\). In order to prove we break it in two steps.

**Step 1** Let \(R\) be a prime ring satisfying one of the conditions (a) and (b) of Theorem 2.1. Then \(R\) has no nilpotent elements of \(R\) form an ideal.

**Proof.** Take an element \(t\) of \(R\) with \(t^2 = 0\) but \(t \neq 0\). Let \(R\) satisfy (a) \((P_1)\) (i). Replacing \(tx\) for \(x\) and \(txt\) for \(y\) and using the fact that \(t^2 = 0\), we have \([[tx]^m, txt] + [(tx)^n, (txt)^m]^k, tx] = [(tx)^m txt, tx] = (tx)^m+3 = 0 \forall x \in R\). Let \(R\) satisfy (a) \((P_1)\) (ii). Replacing \(tx\) for \(x\) and \(txt\) for \(y\) and using the fact that \(t^2 = 0\), we have \([[tx]^m, txt] - [(tx)^n, (tx)^m]^k, tx] = [(tx)^m txt, tx] = (tx)^m+3 = 0 \forall x \in R\).

Next, if \(R\) satisfies (b) \((P_2)\) (i), then by setting \(txt\) for \(x\) and \(xt\) for \(y\), we obtain \([[txt, (xt)^m] + [(xt)^n, (xt)^m]^k, xt] = [txt(xt)^m, xt] = (xt)^m+3 = 0 \forall x \in R\). If \(R\) satisfies (b) \((P_2)\) (ii), then by setting \(txt\) for \(x\) and \(xt\) for \(y\), we obtain \([[txt, (xt)^m] - [(xt)^n, (xt)^m]^k, xt] = [txt(xt)^m, xt] = (xt)^m+3 = 0 \forall x \in R\). If \(tr \neq 0\), then the above obtained result shows that \(tR\) is a nonzero nil right ideal satisfying \(z^{m+3} = 0 \forall z \in tR\).

By an application of a result of [13], rules this out and hence \(tR = 0\), and primeness of \(R\) yields \(t = 0\).

**Step 2** Suppose that \(R\) is a prime ring satisfying one of the conditions (a) and (b) of Theorem 2.1. Then the
commutator ideal of \( R \) is nil.

**Proof.** Let \( R \) satisfy (a). Then we have either (i) \([x^n, y^m] + [x^m, y], x] = 0\), or (ii) \([x^n, y^m] - [x^m, y], x] = 0\). These are polynomial identities with co prime integral coefficients. But no ring of \( p \times p \) commutator ideal of \( R \) is nil. From Step 2, the commutator ideal of \( R \) is nil, in view of Step 1, \( R \) has no nonzero nilpotent elements and hence \( R \) is commutative. This shows (a) \( \implies \) (c).

Using the similar argument as above, one can prove (b) \( \implies \) (c).

**2.0.1. Example**
The ring of \( n \times n \) strictly upper triangular matrices over an associative ring satisfies the hypothesis of Theorem 2.1, but need not be commutative for \( n > 2 \). This shows that Theorem 2.1 is not valid for arbitrary rings.

**3. Commutativity of ring with unity 1**
In what follows, a well known result due to Bell [5] proved commutativity of \( n \)-torsion free ring with unity 1 satisfying the identity \([x^n, y] = [x, y^n]\). In this line of investigation, consider the following ring property:

\((P_3)\): For every \( y \in R \), there exist polynomials \( f(X), g(X), \) and \( h(X) \in \mathbb{Z}[X] \) such that either (i) \( x^n [x, y^s]y^n = f(x)[x^t g(x), y]^t h(x) \) or (ii) \( x^n [x, y^s]y^n = -f(x)[x^t g(x), y]^t h(x) \) where \( m \geq 0, n \geq 0, t \geq 0, s > 1, r > 1 \) are fixed.

Several authors have investigated commutativity of rings satisfying various special cases of this property. The aim of this section is to prove commutativity of ring with unity 1 satisfying \((P_3)\) together with suitable constraint \( Q(n) \). We write \( Q(n) \): For any positive integer \( n \) and \( x, y \in R \), \( n[x, y] = 0 \) implies \([x, y] = 0\).

We begin with the following theorem.

**Theorem 3.1.** Let \( R \) be a ring with unity 1 and \( R \) satisfy the properties \((P_3)\), \( Q(n) \) and at least one of \( p, q \) is zero. Then \( R \) is commutative.

For developing the proof of this theorem we first state the following known results.

**Lemma 3.1.** \([14]\) If \( x, y \in R \) such that \([x, y] \) commutes with \( x \), then for any \( r \geq 1 \), \([x^r, y] = rx^{r-1}[x, y]\).

**Lemma 3.2.** \([3]\) Suppose \( R \) is a ring with unity 1 and \( f : R \rightarrow R \) is any polynomial function of two variables with the property \( f(x + 1, y) = f(x, y) \) for all \( x, y \in R \). If, \( \forall x, y \in R \), \( x^m f(x, y) = 0 \) or \( f(x, y)x^m = 0 \) for a fixed integer \( m \), then necessarily \( f(x, y) = 0 \).

**Lemma 3.3.** \([2]\) Let \( R \) be a ring (may be without unity 1) and for each \( x, y \in R \), there exists a polynomial \( f(X) \in XZ[X] \) such that \([x, y] = [x, y]f(x) \). Then \( R \) is commutative.

Now, we prove the following claims which will be used in proving Theorem 2.1.

**Claim 1**
Let \( R \) be a ring satisfy \((P_3)\) with at least one of \( m, n \) is zero. Then the commutator ideal is nil.
Proof. Putting $y$ by $y + x$ in the both conditions (i) and (ii) of $(P_3)$ and using $(P_3)$, we get

$$x^m[(x + y)^n](y + x)^n = x^m[x, y^n]y^n$$  \hspace{1cm} (1)

This equation is a polynomial identity and if $n = 0$, then we observe that $y = -e_{11} + e_{21}$ and $x = e_{11}$ fail to satisfy the equality in the ring of $2 \times 2$ matrices over $GF(p)$, $p$ a prime. In view of Lemma 1.2, we get the commutator ideal of $R$ is nil. Next, if $m = 0$ in the above equation, then $y = -e_{11} + e_{12}$, and $x = e_{11}$ yields the required result. \hfill \Box

Claim 2

Let $R$ be a ring with unity 1 satisfy $(P_3)$ and $R$ has the property $Q(n)$ with $q = 0$. Then $N(R) \subseteq Z(R)$.

Proof. Let $b \in N(R)$. Then there exists a positive integer $s$ such that

$$b^p \in Z(R), \forall p \geq q, \text{ q is minimal.}$$  \hspace{1cm} (2)

If $q = 1$, then the result is trivial. Suppose that $q > 1$. Putting $y$ by $b^{q-1}$ in both conditions of $(P_3)$, we find either $x^m[x, b^{(q-1)s}b^{q-1}n] = f(x)[x^qg(x), b^{(q-1)}]h(x)$ or $x^m[x, b^{(q-1)s}b^{q-1}n] = -f(x)[x^qg(x), b^{(q-1)}]h(x)$.

In view of (2) and the fact that $(q - 1)s \geq q$, each of the above conditions yields

$$f(x)[x^qg(x), b^{(q-1)}]h(x) = 0.$$  \hspace{1cm} (3)

Again, replacing $y$ by $1 + b^{q-1}$ in $(P_3)$ and using (3), we get

$$x^m[x, (1 + b^{(q-1)s})(1 + b^{q-1})] = 0.$$  \hspace{1cm} (4)

Since $(1 + b^{(q-1)})$ is invertible, (4) becomes

$$x^m[x, (1 + b^{(q-1)s})] = 0, \forall x \in R.$$  \hspace{1cm} (5)

Replace $x$ by $1 + x$ in (5) and use the Lemma 2.3 to get

$$[x, (1 + b^{(q-1)s})] = 0 \forall x \in R.$$

Combining the above equation with the last equation, we find that $[x, 1 + sb^{(q-1)}] = 0$ implies that $s[x, b^{(q-1)}] = 0 \forall x \in R$. By an application of the property $Q(s)$ gives that $[x, b^{(q-1)}] = 0$ and $b^{(q-1)} \in Z(R)$. This contradicts the minimality of $q$ in (2) and hence $q = 1$ and $b \in Z(R)$. \hfill \Box

Proof of Theorem 3.1

Combining Claims 1 and 2, we find

$$C(R) \subseteq N(R) \subseteq Z(R).$$  \hspace{1cm} (6)

Replacing $y$ by $1 + y$ in the property $(P_3)$, we get $x^m[x, (1 + y)^n](1 + y)^n = f(x)[x^qg(x), 1 + y]h(x) = f(x)[x^qg(x), y]h(x) = x^m[x, y^n]y^n$ This implies that $x^m[(x, (1 + y)^n](1 + y)^n - [x, y^n]y^n] = 0$. Putting $x$ by
1 + x in the last equation and using Lemma 2.3, we get \([x, (1 + y)^s](1 + y)^n = [x, y^s]y^n\). By an application of Lemma 2.3 and the above equation, yields \(s(1 + y)^{s+n-1}[x, (1 + y)] = sy^{s+n-1}[x, y]\). This implies that
\[s[x, y](1 + y)^{s+n-1} - y^{s+n-1} = 0.\] (7)

Using the property \(Q(s)\) in last equation, we get \([x, y](1 + y)^{s+n-1} - y^{s+n-1} = 0\). This polynomial identity can be re-written in the form \([x, y] = [x, y]h(y)\) for some \(h(X) \in XZ[X]\). By an application of Lemma 2.4, it gives the required result.

The following are immediate consequence of the above theorem.

**Corollary 3.1.** \([15]\) Let \(R\) be a ring with unity 1 in which for all \(x, y \in R\) there exist integers \(n \geq 0, s \geq 0, t \geq 0, p \geq 0, q \geq 0, r > 0, m > 1\) such that \(x^s[x, y]x^t = \pm y^p[x^n, y^m]y^q\). Then \(R\) is commutative if at least one of \(p\) and \(q\) is zero.

**Corollary 3.2.** \([5]\) Let \(R\) be a \(n\)-torsion free ring with unity 1 and \(n > 1\) be a fixed positive integer. Let \(R\) satisfy \([x^n, y] = [x, y^n]\). Then \(R\) is commutative.

**Corollary 3.3.** Let \(R\) be a ring with unity 1 in which for each \(y \in R\) there exist integers \(m\) and \(f(t), g(t), h(t) \in Z[t]\) such that \(x^m[x, y] = f(x)[x^2g(x), y]h(x)\). Then \(R\) is commutative (and conversely).

**Remark 3.1.** The following example strengthens the existence of the property \(Q(n)\) in the hypothesis of Theorem 3.1.

**Example 3.1.** Take \(R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in GF(2) \right\}\). Then \(R\) is a non commutative ring with unity satisfying the property \([x^2, y] = [x, y^2]\).

**Remark 3.2.** From the above example one can observe that Theorem 3.1 cannot be generalized to arbitrary ring.

4. Applications of commutativity theorems in near-rings

A beautiful result was proved by Bell [4] as \(N\) is a distributively generated \((d \ g)\) near-ring with an identity, and if for each \(x, y \in N\), there exists \(n > 1\), depending on \(x\) and \(y\), such that \((xy - yx)^n = xy - yx\), then \(N\) is a commutative ring. He also pointed out that the conclusion will not hold if one drops the existence of an identity. Later Bell weakened this identity for \((d \ g)\) periodic near-ring with identity 1 and also Ligh [16] extended it without identity. In this line of investigation, we consider the following properties:

\((P_4)\) For each \(x, y \in N\), there exist \(m = m(x, y) > 1, n = n(x, y) > 1\) and \(k > 0\) with \([x^m, y^n]^k = [x^m, y^n]\)

\((P_5)\) For each \(x, y \in N\), there exist \(m = m(x, y) > 1, n = n(x, y) > 1\) such that \((x^m y - yx^m)^n = x^m y - yx^m\).

We begin with

**Theorem 4.1.** Let \(N\) be a \(D\)-near-ring satisfying \((P_4)\). If \(N^* \subseteq Z(N)\), then \(N \mid N^*\) is periodic and commutative ring.
Theorem 4.2. Let $N$ be a $(d, g)$ near-ring with no nonzero divisors of zero and for each $x, y \in N$. Let $N$ satisfy $(P_5)$. Then $R$ is commutative.

In order to prove above Theorems 4.1 and 4.2, we need the following lemmas.

Lemma 4.1 ([18]). If $N$ is a zero commutative near-ring, then $N^*$ is an ideal of $N$.

Lemma 4.2 ([8]). Let $N$ be a zero symmetric near-ring satisfying the following properties: (i) for each $x \in N$, there exists a positive integer $n(x)$, 1, such that $x^{n(x)} = x$ (ii) every nontrivial homomorphic image of $N$ contains a nonzero central idempotent. Then $(N, +)$ is commutative.

Lemma 4.3 ([8]). Let $N$ be a near-ring with no nonzero nilpotent elements. Then $N$ contains a family of completely prime ideals with trivial intersection.

Lemma 4.4 ([10]). Let $N$ be a $(d, g)$ near-ring such that $(N, +)$ is abelian. Then $N$ is a ring.

The following lemma can be proving by slight modification of of Bell [6].

Lemma 4.5. Let $N$ be a $(d, g)$ near-ring with no nonzero divisors of zero. If for each $x, y \in N$, there exists $m > 1$ and $n > 1$ depending on $x$ and $y$ such that $[x^m, y]^n = [x^m, y]$, then the set of nilpotent elements is an ideal of $N$.

Now, we prove the main theorems.

Proof of Theorem 4.1
Since $N^* \subseteq Z(N)$, as $N^*$ is an ideal of $N$, by Lemma 4.1. Consider the near-ring $N | N^*$. Take a nonzero distributive element $x$ of $N | N^*$ (where $N \mid N^*$ can be written as a sub direct product of near-rings without zero divisors, so one may assume $N \mid N^*$ has no nonzero divisors). It is observed that $x = xf(x)$ for some $f(x) \in N$ by property $(P_4)$ when $y = x$. This implies that $e = f(d)$ is a nonzero idempotent. Since $e(\nu - r) = 0$ for every $r \in N \mid N^*$, so $e$ is the left identity in $N \mid N^*$. Taking arbitrary $a, b \in N \mid N^*$ and using the fact that $e$ commutes with $x$ and $x = xe$, we have $0 = (a + b)xe - (axe - bxe) = [(a + b)e - ae - be]x$. Thus, $e$ is multiplicative identity in $N \mid N^*$, because $N \mid N^*$ has no nonzero divisors, $e$ is the distributive element of $N \mid N^*$ of $N \mid N^*$. Let $a$ be an arbitrary nonzero element of $N \mid N^*$. Then $a = f(a)$ where $f(a) \in N$ by Lemma 4.2, we have $f(a) = a f'(a)$ for some $f'(a) \in N$. Now, $a = a^2 f'(a)$, gives $a f'(a) = 1$. This implies that $N \mid N^*$ is a division near-ring. Thus, $N \mid N^*$ is additively commutative and is a ring. Hence, $N \mid N^*$ is periodic by Chacron’s criterion [9] and a commutative ring by [[4], Theorem 2].

Proof of Theorem 4.2
If there exist $x$ and $y$ such that $(x^m y - y x^m) \not= 0$, $(x^m y - y x^m)^{n-1} = e$ is a nonzero idempotent. But $N$ has no nonzero divisors of zero and is $(d, g)$, one can observe that if $w \not= 0$ is a right distributive element of $N$ and $r$ is arbitrary, then $e(\nu - r) = 0$ implies $er = r$ and $(re - r)w = 0$ implies $re = r$. Hence, $e$ is an identity of $N$ and it’s nonzero idempotent.

Now we claim that $(N, +)$ is abelian by demonstrating that either every element of $(N, +)$ is of order two or $N$ is an $m$-system. Let $x, y$ and $z$ be in $N$ such that $xz = yz$ and $z \not= 0$. If $z$ is central then $xz = zy$ implies $x = y$. If $z$ is not central, then there exists $t \not= 0$ such that $(zt - tz)t \not= 0$. Thus $zt(zt - tz) \not= 0$ implies
that there exists \( m > 1 \) such that \( [z^m(zt) - (zt)z^m]^{n-1} = e \). But \( z \) has a right inverse \( z' \). Hence \( xz = yz \) implies that \( x = xzz' = yzz' = y \). Take \( e + e \neq 0 \) and is not central. Then from above, \( e + e \) has a right inverse \( h \). Since \( N \) has no nonzero divisors of zero, \( h \) is also a left inverse. So \( h(e + e) = e \) implies that \( he + he = e \). Hence, \( N \) is an \( m \)-system and \( (N, +) \) is abelian by Lemma 4.5. Suppose \( e + e \neq 0 \) and is central, then \( e + e \) is also right distributive. Let \( a, b \) be in \( N \). Expanding \( (a + b)(e + e) \) by using both distributive laws, we get \( a + b = b + a \). Again, \( (N, +) \) is abelian. Now by Lemma 4.4, \( N \) is a ring.

In view of [4] it was shown that every element of a commutative (distributive) near-ring \( N \) is either a zero divisor or \( N \) is a ring.

Thus we obtain the following corollaries.

**Corollary 4.1 ([16]).** A \((d \; g)\) near-ring \( N \) is commutative if and only if \( , \) for each \( x, y \in N \) there exists \( m > 1 \), depending on \( x \) and \( y \), such that \( (xy - yx)^n = xy - yx \).

**Corollary 4.2 ([8]).** Let \( N \) be a \((d \; g)\) near-ring with 1 and for each \( x, y \in N \), let there be \( n(x, y) > 1 \), such that \( (xy - yx)^n = xy - yx \). Then \( N \) is a commutative ring.

### 5. Open problems

5.1 Let \( R \) be a left (resp. right) \( s \)-unital ring in which for every \( x, y \in R \) there exist polynomials \( f(X), g(X), h(X) \in Z(X) \) such that \( x^p[x^n, y]y^q = f(y)[x^p g(x), y]h(y) \), where \( p \geq 0, q \geq 0, t \geq 0, n > 1, k > 1 \) are fixed. If \( R \) satisfies the property \( Q(n) \), then \( R \) is commutative.

5.2 Let \( N \) be a \( D \)-near-ring satisfying the condition \( [x, y] = [x^n, y^m]p(x, y) \), where \( p(x, y) \) denotes an element of a near-ring which is the finite sum of powers of \( x^t, t \geq 2 \), for each pair of elements \( x, y \in N \), there exist integers \( n = n(x, y) \geq 1 \) and \( m = m(x, y) \geq 1 \). If idempotent elements of \( N \) are central, then \( N \) is commutative.

5.3 Suppose \( N \) is a \((d \; g)\) near-ring satisfying the condition \( [x, y] = [x^n, y^m]p(x, y) \), where \( p(x, y) \) is an element of a near-ring with finite sum of powers of \( x^t, t \geq 2 \), for each pair of elements \( x, y \in N \), there exist integers \( n = n(x, y) \geq 1 \) and \( m = m(x, y) \geq 1 \). Then \( N \) is commutative.

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