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Commutativity theorems for rings with its applications

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Abstract: The present paper is to prove new commutativity theorems for rings (see Theorems 2.1, and 3.1). In addition, applications of commutativity theorems for rings to near-rings, we investigate some polynomial identities (P_4) and (P_5) with exponents depending on x and y for a certain class of near-rings and then, under some constraints, it is shown that D-near-ring as well as distributively generated $(d\ g)$ near-rings are commutative. Finally, we close our discussion with some open problems.

Key words: Commutativity, distributively generated $(d\ g)$ near-ring, D-near-ring, ring with identity, semiprime ring

1. Introduction

Throughout, R will be an associative ring (may be without unity 1), $Z(R)$ will represent the centre of R, $N(R)$ the set of all nilpotent elements in R, and $C(R)$ the commutator ideal of R. For any $x, y \in R$, the commutator $xy - yx$ will be denoted by $[x, y]$. By $GF(q)$, we mean the Galois field (finite field) with q elements, and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$, and $Z[X]$ is the totality of polynomials in X with coefficients in Z, the ring of integers. A classical result in ring theory is the theorem of Herstein [13] , which gives a condition on the commutators $xy–yx$ of a ring that is necessary and sufficient for commutativity, and established the commutativity of R if, for each $x, y \in R$ there exists a natural number $n = n(x, y) > 1$ depending on x and y, and R satisfies the condition $[x, y]^n = [x, y]$. Harmanci [12] proved the commutativity of a ring with unity 1 satisfying $[x^k, y] = [x, y^k]$, $k = n, n+1$. Further, Gupta [11] extended it for the commutativity of a semiprime ring R with unity 1 when R satisfies $[x^m, y] - [x, y^m] \in Z(R)$ and $[x^{m+1}, y] - [x, y^{m+1}] \in Z(R)$, for all $x, y \in R$. It is a natural question: "Whether the existence of unity 1 in the semiprime ring of Gupta's [11] result is essential with both the identities are necessary?" Answering this question, for any $x, y \in R$, and k and n are natural numbers, we investigate the ring properties:

(P_1) There exists a positive integer $m > 1$ such that either (i) $[[x^n, y^m]^k + [x^m, y], x] = 0$ or (ii) $[[x^n, y^m]^k - [x^m, y], x] = 0$.

(P_2) There exists a positive integer $m > 1$ such that either (i) $[[x^n, y^m]^k + [x^m, y], y] = 0$ or (ii) $[[x^n, y^m]^k - [x^m, y], y] = 0$,

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In addition, it is shown that the commutativity of semiprime rings without unity satisfying one of the conditions (P_1) and (P_2).

In Section 2, we first prove the conditions (P_1) and (P_2) for semi prime rings that is an extension of the results of [1] and [11], and provide one example to show that this result can not extend for arbitrary rings. Section 3 is devoted to commutativity of rings with unity 1 with later defined properties (P_3) and $Q(n)$, and it demonstrates an example that the hypotheses are not superfluous. Section 4 includes some applications of commutativity theorems in near-rings as; "contributions on commutativity of D-near-rings" as well as "distributively generated $(d\ g)$ near-rings". Finally, we conclude it with some open problems in Section 5.

2. A commutativity theorem for semiprime rings

We begin with the following theorem.

Theorem 2.1. *Let R be a semiprime ring. Then the following statements are equivalent.*

a) *R satisfies (P_1).*

b) *R satisfies (P_2).*

c) *R is a commutative ring.*

Before proving Theorem 2.1, we need the following result.

Lemma 2.1. *[6] Let R be a ring satisfying an identity $q(X) = 0$, where $q(X)$ is a polynomial in a finite number of noncommuting indeterminates, its coefficients being integers with highest common factor 1. If there exist no prime p for which the ring 2×2 matrices over $GF(p)$ satisfies $q(X) = 0$, then R has a nil commutator ideal and the nilpotent elements of R form an ideal.*

Proof of Theorem 2.1

Since a semiprime ring R is isomorphic to a sub direct sum of prime rings R_α, each of α, which as a homomorphic image of R satisfies the hypothesis placed on R, we may assume that the ring R is prime satisfying one of the conditions (a) and (b). Given R satisfies (c), then (a) and (b) follows immediately.

Next, we claim that (a) $\Longrightarrow$ (c) and (b) $\Longrightarrow$ (c). In order to prove we break it in two steps.

Step 1 Let R be a prime ring satisfying one of the conditions (a) and (b) of Theorem 2.1. Then R has no nilpotent elements of R form an ideal.

Proof. Take an element t of R with $t^2 = 0$ but $t \neq 0$. Let R satisfy (a) (P_1) (i). Replacing tx for x and txt for y and using the fact that $t^2 = 0$, we have $[[(tx)^m, txt] + [(tx)^n, (txt)^m]^k, tx] = [(tx)^m txt, tx] = (tx)^{m+3} = 0\ \forall x \in R$. Let R satisfy (a) (P_1) (ii). Replacing tx for x and txt for y and using the fact that $t^2 = 0$, we have $[[(tx)^m, txt] - [(tx)^n, (txt)^m]^k, tx] = [(tx)^m txt, tx] = (tx)^{m+3} = 0\ \forall x \in R$.

Next, if R satisfies (b) (P_2) (i)., then by setting txt for x and xt for y, we obtain $[[txt, (xt)^m] + [(txt)^n, (xt)^m]^k, xt] = [txt(xt)^m, xt] = (tx)^{m+3} = 0\ \forall x \in R$. if R satisfies (b) (P_2) (ii)., then by setting txt for x and xt for y, we obtain $[[txt, (xt)^m] - [(txt)^n, (xt)^m]^k, xt] = [txt(xt)^m, xt] = (tx)^{m+3} = 0\ \forall x \in R$. If $tR \neq 0$, then the above obtained result shows that tR is a nonzero nil right ideal satisfying $z^{m+3} = 0\ \forall z \in tR$. By an application of a result of [13], rules this out and hence $tR = 0$, and primeness of R yields $t = 0$.

Step 2 Suppose that R is a prime ring satisfying one of the conditions (a) and (b) of Theorem 2.1.Then the

commutator ideal of R is nil.
Proof. Let R satisfy (a).Then we have either (i) $[[x^n, y^m]^k + [x^m, y], x] = 0$, or (ii) $[[x^n, y^m]^k - [x^m, y], x] = 0$. These are polynomial identities with co prime integral coefficients. But no ring of 2×2 matrices over $GF(p)$, p a prime, satisfies one of the above properties (a) (P_1) (i) or (P_1) (ii) by taking $x = e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

In view of Lemma 2.1 the commutator ideal of R is nil. From Step 2, the commutator ideal of R is nil, in view of Step 1, R has no nonzero nilpotent elements and hence R is commutative. This shows (a) $\Longrightarrow$ (c). Using the similar argument as above, one can prove (b) $\Longrightarrow$ (c).

2.0.1. Example

The ring of $n \times n$ strictly upper triangular matrices over an associative ring satisfies the hypothesis of Theorem 2.1, but need not be commutative for $n > 2$. This shows that Theorem 2.1 is not valid for arbitrary rings.

3. Commutativity of ring with unity 1

In what follows, a well known result due to Bell [5] proved commutativity of n-torsion free ring with unity 1 satisfying the identity $[x^n, y] = [x, y^n]$. In this line of investigation, consider the following ring property:
(P_3): For every $y \in R$, there exist polynomials $f(X), g(X)$, and $h(X) \in Z[X]$ such that either (i) $x^m[x, y^s]y^n = f(x)[x^t g(x), y]^r h(x)$ or (ii) $x^m[x, y^s]y^n = -f(x)[x^t g(x), y]^r h(x)$ where $m \geq 0$, $n \geq 0$, $t \geq 0$, $s > 1$, $r > 1$ are fixed.

Several authors have investigated commutativity of rings satisfying various special cases of this property. The aim of this section is to prove commutativity of ring with unity 1 satisfying (P_3) together with suitable constraint $Q(n)$. We write $Q(n)$: For any positive integer n and $x, y \in R$, $n[x, y] = 0$ implies $[x, y] = 0$.

We begin with the following theorem.

Theorem 3.1. *Let R be a ring with unity 1 and R satisfy the properties (P_3), $Q(n)$ and at least one of p, q is zero. Then R is commutative.*

For developing the proof of this theorem we first state the following known results.

Lemma 3.1. *[14] If $x, y \in R$ such that $[x, y]$ commutes with x, then for any $r \geq 1$, $[x^r, y] = rx^{r-1}[x, y]$.*

Lemma 3.2. *[3] Suppose R is a ring with unity 1 and $f : R \longrightarrow R$ is any polynomial function of two variables with the property $f(x + 1, y) = f(x, y)$ for all $x, y \in R$. If, $\forall x, y \in R$, $x^m f(x, y) = 0$ or $f(x, y)x^m = 0$ for a fixed integer m, then necessarily $f(x, y) = 0$.*

Lemma 3.3. *[2] Let R be a ring (may be without unity 1) and for each $x, y \in R$, there exists a polynomial $f(X) \in XZ[X]$ such that $[x, y] = [x, y]f(x)$. Then R is commutative.*

Now, we prove the following claims which will be used in proving Theorem 2.1.

Claim 1

Let R be a ring satisfy (P_3) with at least one of m, n is zero. Then the commutator ideal is nil.

Proof. Putting y by $y+x$ in the both conditions (i) and (ii) of (P_3) and using (P_3), we get

$$x^m[x,(y+x)^s](y+x)^n = x^m[x,y^s]y^n \tag{1}$$

This equation is a polynomial identity and if $n=0$, then we observe that $y=-e_{11}+e_{21}$ and $x=e_{11}$ fail to satisfy the equality in the ring of 2×2 matrices over $GF(p)$, p a prime. In view of Lemma 1.2, we get the commutator ideal of R is nil. Next, if $m=0$ in the above equation, then $y=-e-11+e_{12}$, and $x=e_{11}$ yields the required result. □

Claim 2

Let R be a ring with unity 1 satisfy (P_3) and R has the property $Q(n)$ with $q=0$. Then $N(R)\subseteq Z(R)$.

Proof. Let $b\in N(R)$. Then there exists a positive integer s such that

$$b^p \in Z(R),\ \forall p\geq q,\ \text{q is minimal.} \tag{2}$$

If $q=1$, then the result is trivial. Suppose that $q>1$. Putting y by b^{q-1} in both conditions of (P_3), we find either $x^m[x,b^{(q-1)s}]b^{(q-1)n} = f(x)[x^tg(x),b^{(q-1)}]^rh(x)$ or $x^m[x,b^{(q-1)s}]b^{(q-1)n} = -f(x)[x^tg(x),b^{(q-1)}]^rh(x)$. In view of (2) and the fact that $(q-1)s\geq q$, each of the above conditions yields

$$f(x)[x^tg(x),b^{(q-1)}]^rh(x)=0. \tag{3}$$

Again, replacing y by $1+b^{q-1}$ in (P_3) and using (3), we get

$$x^m[x,(1+b^{(q-1)})^s](1+b^{q-1})^n=0. \tag{4}$$

Since $(1+b^{(q-1)})$ is invertible, (4) becomes

$$x^m[x,(1+b^{(q-1)})^s]=0,\ \forall x\in R. \tag{5}$$

Replace x by $1+x$ in (5) and use the Lemma 2.3 to get

$$[x,(1+b^{(q-1)})^s]=0\ \forall x\in R.$$

Combining the above equation with the last equation, we find that $[x,1+sb^{(q-1)}]=0$ implies that $s[x,b^{(q-1)}]=0\ \forall x\in R$. By an application of the property $Q(s)$ gives that $[x,b^{(q-1)}]=0$ and $b^{(q-1)}\in Z(R)$. This contradicts the minimality of q in (2) and hence $q=1$ and $b\in Z(R)$. □

Proof of Theorem 3.1

Combining Claims 1 and 2, we find

$$C(R)\subseteq N(R)\subseteq Z(R). \tag{6}$$

Replacing y by $1+y$ in the property (P_3), we get $x^m[x,(1+y)^s](1+y)^n = f(x)[x^tg(x),1+y]^rh(x) = f(x)[x^tg(x),y]^rh(x) = x^m[x,y^s]y^n$ This implies that $x^m([x,(1+y)^s](1+y)^n-[x,y^s]y^n)=0$. Putting x by

$1+x$ in the last equation and using Lemma 2.3, we get $[x,(1+y)^s](1+y)^n = [x,y^s]y^n$ By an application of Lemma 2.3 and the above equation, yields $s(1+y)^{s+n-1}[x,(1+y)] = sy^{s+n-1}[x,y]$. This implies that

$$s[x,y](1+y)^{s+n-1} - y^{s+n-1} = 0. \tag{7}$$

Using the property $Q(s)$ in last equation, we get $[x,y](1+y)^{s+n-1} - y^{s+n-1}) = 0$. This polynomial identity can be re-written in the form $[x,y] = [x,y]h(y)$ for some $h(X) \in XZ[X]$. By an application of Lemma 2.4, it gives the required result.

The following are immediate consequence of the above theorem.

Corollary 3.1. *[15] Let R be a ring with unity 1 in which for all $x,y \in R$ there exist integers $n \geq 0$, $s \geq 0, t \geq 0, p \geq 0, q \geq 0, r > 0, m > 1$ such that $x^s[x,y]x^t = \pm\, y^p[x^n,y^m]^r y^q$. Then R is commutative if at least one of p and q is zero.*

Corollary 3.2. *[5] Let R be a n-torsion free ring with unity 1 and $n > 1$ be a fixed positive integer. Let R satisfy $[x^n,y] = [x,y^n]$. Then R is commutative.*

Corollary 3.3. *Let R be a ring with unity 1 in which for each $y \in R$ there exist integers m and $f(t), g(t), h(t) \in Z[t]$ such that $x^m[x,y] = f(x)[x^2g(x),y]h(x)$. Then R is commutative (and conversely).*

Remark 3.1. *The following example strengthens the existence of the property $Q(n)$ in the hypothesis of Theorem 3.1.*

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Example 3.1. *Take $R = \left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix} \;\middle|\; \alpha,\beta,\gamma,\delta \in GF(2) \right\}$. Then R is a non commutative ring with unity satisfying the property $[x^2,y] = [x,y^2]$.*

Remark 3.2. *From the above example one can observe that Theorem 3.1 cannot be generalized to arbitrary ring.*

4. Applications of commutativity theorems in near-rings

A beautiful result was proved by Bell [4] as N is a distributively generated $(d\ g)$ near-ring with an identity, and if for each $x,y \in N$, there exists $n > 1$, depending on x and y, such that $(xy-yx)^n = xy-yx$, then N is a commutative ring. He also pointed out that the conclusion will not hold if one drops the existence of an identity. Later Bell weakened this identity for $(d\ g)$ periodic near-ring with identity 1 and also Ligh [16] extended it without identity. In this line of investigation, we consider the following properties:

(P_4) For each $x,y \in N$, there exist $m = m(x,y) > 1$, $n = n(x,y) > 1$ and $k > 0$ with $[x^m,y^n]^k = [x^m,y^n]$

(P_5) For each $x,y \in N$, there exist $m = m(x,y) > 1$, $n = n(x,y) > 1$ such that $(x^my - yx^m)^n = x^my - yx^m$.

We begin with

Theorem 4.1. *Let N be a D-near-ring satisfying (P_4). If $N^* \subseteq Z(N)$, then $N \mid N^*$ is periodic and commutative ring.*

Theorem 4.2. *Let N be a $(d\ g)$ near-ring with no nonzero divisors of zero and for each $x, y \in N$. Let N satisfy (P_5). Then R is commutative.*

In order to prove above Theorems 4.1 and 4.2, we need the following lemmas.

Lemma 4.1 ([18]). *If N is a zero commutative near-ring, then N^* is an ideal of N.*

Lemma 4.2 ([8]). *Let N be a zero symmetric near-ring satisfying the following properties: (i) for each $x \in N$, there exists a positive integer $n(x).1$, such that $x^{n(x)} = x$ (ii) every nontrivial homomorphic image of N contains a nonzero central idempotent. Then $(N, +)$ is commutative.*

Lemma 4.3 ([8]). *Let N be a near-ring with no nonzero nilpotent elements. Then N contains a family of completely prime ideals with trivial intersection.*

Lemma 4.4 ([10]). *Let N be a $(d,\ g)$ near-ring such that $(N, +)$ is abelian. Then N is a ring.*

The following lemma can be proving by slight modification of of Bell [6].

Lemma 4.5. *Let N be a $(d\ g)$ near-ring with no nonzero divisors of zero. If for each $x, y \in N$, there exists $m > 1$ and $n > 1$ depending on x and y such that $[x^m, y]^n = [x^m, y]$, then the set of nilpotent elements is an ideal of N.*

Now, we prove the main theorems.

Proof of Theorem 4.1

Since $N^* \subseteq Z(N)$, as N^* is an ideal of N, by Lemma 4.1. Consider the near-ring $N \mid N^*$. Take a nonzero distributive element x of $N \mid N^*$ (where $N \mid N^*$ can be written as a sub direct product of near-rings without zero divisors, so one may assume $N \mid N^*$ has no nonzero divisors). It is observed that $x = xf(x)$ for some $f(x) \in < x >$ by property (P_4) when $y = x$. This implies that $e = f(d)$ is a nonzero idempotent. Since $e(er - r) = 0$ for every $r \in N \mid N^*$, so e is the left identity in $N \mid N^*$. Taking arbitrary $a, b \in N \mid N^*$ and using the fact that e commutes with x and $x = xe$, we have $0 = (a+b)xe - (axe - bxe) = [(a+b)e - ae - be]x$. Thus, e is multiplicative identity in $N \mid N^*$, because $N \mid N^*$ has no nonzero divisors, e is the distributive element of $N \mid N^*e$. Let a be an arbitrary nonzero element of $N \mid N^*$. Then $a = f(a)$ where $f(a) \in < a >$. Using Lemma 4.2, we have $f(a) = af'(a)$ for some $f'(a) \in < 1a >$. Now, $a = a^2 f'(a)$, gives $af'(a) = 1$. This implies that $N \mid N^*$ is a division near-ring. Thus, $N \mid N^*$ is additively commutative and is a ring. Hence, $N \mid N^*$ is periodic by Chacron's criterion [9] and a commutative ring by [[4], Theorem 2].

Proof of Theorem 4.2

If there exist x and y such that $(x^m y - yx^m) \neq 0$, $(x^m y - yx^m)^{n-1} = e$ is a nonzero idempotent. But N has no nonzero divisors of zero and is $(d\ g)$, one can observe that if $w \neq 0$ is a right distributive element of N and r is arbitrary, then $e(er - r) = 0$ implies $er = r$ and $(re - r)w = 0$ implies $re = r$. Hence, e is an identity of N and it's nonzero idempotent.

Now we claim that $(N, +)$ is abelian by demonstrating that either every element of $(N, +)$ is of order two or N is an m-system. Let x, y and z be in N such that $xz = yz$ and $z \neq 0$. If z is central then $zx = zy$ implies $x = y$. If z is not central, then there exists $t \neq 0$ such that $(zt - tz)t \neq 0$. Thus $z(zt - tz) \neq 0$ implies that there exists $m > 1$ such that $[z^m(zt) - (zt)z^m]^{n-1} = e$. But z has a right inverse z'. Hence $xz = yz$ implies that $x = xzz' = yzz' = y$. Take $e+e \neq 0$ and is not central. Then from above, $e+e$ has a right inverse h. Since N has no nonzero divisors of zero, h is also a left inverse. So $h(e+e) = e$ implies that $he + he = e$. Hence, N is an m-system and $(N, +)$ is abelian by Lemma 4.5. Suppose $e + e \neq 0$ and is central, then $e + e$ is also right distributive. Let a, b be in N. Expanding $(a + b)(e + e)$ by using both distributive laws, we get $a + b = b + a$. Again, $(N, +)$ is abelian. Now by Lemma 4.4, N is a ring.

In view of [4] it was shown that every element of a commutative (distributive) near-ring N is either a zero divisor or N is a ring.

Thus we obtain the following corollaries.

Corollary 4.1 ([16])**.** *A $(d\ g)$ near-ring N is commutative if and only if , for each $x, y \in N$ there exists $m > 1$, depending on x and y, such that $(xy - yx)^n = xy - yx$.*

Corollary 4.2 ([8])**.** *Let N be a $(d\ g)$ near-ring with 1 and for each $x, y \in N$, let there be $n(x, y) > 1$, such that $(xy - yx)^n = xy - yx$. Then N is a commutative ring.*

5. Open problems

5.1 Let R be a left (resp. right) s-unital ring in which for every $x, y \in R$ there exist polynomials $f(X), g(X), h(X) \in Z(X)$ such that $x^p[x^n, y]y^q = f(y)[x^t g(x), y]^k h(y)$, where $p \geq 0, q \geq 0, t \geq 0, n > 1, k > 1$ are fixed. If R satisfies the property $Q(n)$,then R is commutative.

5.2 Let N be a D-near-ring satisfying the condition $[x, y] = [x^n, y^m]p(x, y)$, where $p(x, y)$ denotes an element of a near-ring which is the finite sum of powers of $x^t, t \geq 2$, for each pair of elements $x, y \in N$, there exist integers $n = n(x, y) \geq 1$ and $m = m(x, y) \geq 1$. If idempotent elements of N are central, then N is commutative.

5.3 Suppose N is a $(d\ g)$ near-ring satisfying the condition $[x, y] = [x^n, y^m]p(x, y)$, where $p(x, y)$ is an element of a near-ring with finite sum of powers of $x^t, t \geq 2$, for each pair of elements $x, y \in N$, there exist integers $n = n(x, y) \geq 1$ and $m = m(x, y) \geq 1$. Then N is commutative.

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