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A kind of fixed point theorem on the complete C*-Algebra valued s-Metric spaces

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Abstract: The area of metric space is a widely used technique and this concept allows us to generalize the notion of continuity. The discrete metric is used for a number of purposes, principally to test properties that we suspect may hold for all metric spaces. Moreover, this method is extensively applied in optimization, approximation theory, engineering, dynamical systems, computer sciences and applied mathematics. Stefan Banach, who is a renowned mathematician, constructed a new result called by "Banach's fixed point theorem or the Banach contraction principle" in 1922.

Banach's fixed point theory serves as an essential tool for various branches of mathematical analysis and its applications. Results of the fixed point theory in different metric spaces have been very interesting and can be applied to real life which is why a study on this topic is very significant and useful. Variety of examples have been provided to understand and appreciate the beauty of metric spaces.

The purpose of the paper is to analyse the C^*-algebra valued s- metric space using some basic properties of s- metric space. Some fixed point results have been demonstrated and extended on the complete C^*- algebra valued s-metric space. Also, the existence and uniqueness of the fixed point theories on that space with the Banach contraction principle have been demonstrated. At the end of the paper, several numerical examples have been provided to support our results.

Key words: Fixed Points, Banach Contraction Mapping, C^*-Algebra, Complete s- Metric Space, Fixed Point Theory.

1. Introduction

A metric space is just a set X equipped with a function d of two variables which measures the distance between points; $d(x, y)$ is the distance between two points x and y in X. It turns out if we put mild and natural conditions on the function d, we can develop a general notion of distance. So it covers distances between numbers, vectors, sequences, functions, sets and much more. Within this theory, we can formulate and prove results about convergence and continuity once and for all.

The purpose of this research paper is to define and develop the C^*-algebra valued s- metric space using some basic properties of s- metric space. Sedghi et al. introduced s-metric space in 2011, see [16]. The s-metric space is an extension of D^*-metric space and given with three dimensions. Since 2011, many mathematicians have worked on this metric space with different types of fixed point results, see [1, 2, 14, 15]. In [11–13], Özer and Omran considered valued metric spaces and proved some types of fixed point theories in different kind of metric spaces. Many C^*-algebra results have been proved, for the fixed point problem in generalized metric space, like [3].

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This notion is also the generalization of G-Metric space [9] and D^*-metric space [14]. For this metric space, Sedghi and Shobe [15], proved some properties and some fixed point theorems by considering a self-map on the such space. Further, Jiang and Sun [7], studied on the mapping for C^*-algebra valued metric spaces and introduced some fixed point theorems on them. They proposed a new type of metric space and also gave Fixed Point Theorems relation with self- maps contractive or expansive conditions. Later on, The existence of fixed points of self-mappings satisfying a contractive type condition were worked by a lot of mathematicians.

The idea of C^*-algebra was introduced, by Douglas, see [4–6] and also, Ma and Jiang [8], established the notion of C^*- algebra valued metric spaces, and proved some fixed point theorems for contractive and expansive mappings.

Aim of the paper is to present the concept of C^*-algebra valued s- metric space. Firstly, we define C^*-algebra valued s- metric space using some basic properties of s- metric space. Then, we introduce fixed points on the complete C^*- algebra valued s-metric space. Also, we demonstrate the existence and uniqueness of the fixed point theorems on that space with the Banach contraction principle. Last but not least, we give some numerical examples to promote our results.

2. Preliminaries

2.1. C^*- Algebra

Definition 2.1. Let S be a unital algebra with the unit I. An involution in S is an operation $*$ on S (or, a transformation $* : S \to S,\ x \to x^*$) satisfying the following conditions:
1- For all x in S, $(x^*)^* = x$.
2- For all $x,\ y$ in S we have $(xy)^* = y^* x^*$.
Then the pair (S,*) is said to be a *-Algebra.

Definition 2.2. A C^*- Algebra K is a Banach algebra over C (the field of complex numbers) with a map $k \to k^*$, for $k \in K$ which satisfies the following properties:

1. It is an involution that $k^{**} = k,\ \forall\ k \in K$.
2. For all $k_1,\ k_2$ in K, we have $(k_1 + k_2) = k_1^* + k_2^*$ and $(k_1 k_2)^* = k_2^* k_1^*$.
3. For every complex number μ and every k_1 in K, we have
$(\mu k_1)^* = \bar{\mu} k_1^*$.
4. For all k_1 in K, we have $\|k_1 k_2^*\| = \|k_1\| \|k_2^*\|$. (operator norm)

Remark 2.1. *The first three identities say that K is K^*-algebra. But the last identity is called the C^*- identity and is equivalent to*

$$\|k_1 k_1^*\| = \|k_1\|^2.$$

Definition 2.3. Let K_1 and K_2 be C^*- Algebras. Then the mapping $\mu : K_1 \to K_2$ is a C^*-Homomorphism if the following conditions are satisfied :
i. $\mu(k_1^*) = (\mu(k_1))^*$
ii. $\mu(k_1 k_2) = \mu(k_1)\mu(k_2)$
iii. $\mu(k_1 + k_2) = \mu(k_1) + \mu(k_2)$
for all $k_1,\ k_2$ in K.

Remark 2.2. *We can say C^*- identity(in the remark 2.1) if any homomorphism between C^*- Algebras is bounded and its norm is less than or equal to 1.*

Example 2.1. *(1) Assume $M_n(C)$ be a set of all square matrices $n \times n$ over C with the involution conjugate transpose. Then $M_n(C)$ is C^*- Algebra.*

(2) Supposing that H be a Hilbert space. Then a B(H) (which is a collection of all bounded operators, with γ^* is the dual of the operator $\gamma : H \to H$,) is C^*- Algebra.

2.2. Positivity

Definition 2.4. Assume K is (unital) C^*- Algebra, k in K and k is self adjoint ($k^* = k$). Then k is called positive element if $\sigma(k)$(spectrum of k) is a positive real number.

Remark 2.3. *Let k be a positive element. Then we can denote it by $k \geqslant 0$ and we will assume $K_+ = \{k \in K : k \geqslant 0\}$ and called a positive set of K.*

Lemma 2.1. *Let us suppose that V be a C^*- Algebra. Then the following statements are true;*
1. If $k_1 \in K$ is a normal, then $k_1 k_1^* \geqslant 0$.
2. If $k_1 \in K$ is a self adjoint and $\|k_1\| \leqslant 1$, then $k_1 \geqslant 0$.
3. If $k_1,\ k_2 \in K_+$, then $k_1 + k_2 \in K_+$.
4. K_+ is closed in K.

Remark 2.4. *Let $k_1,\ k_2 \in K_+$. Then it can be defined $k_1 \leqslant k_2$ for $k_2 - k_1 \geqslant 0$. So, this implies that $(K_+, \leqslant)$ is a partially ordered relation.*

2.3. s-Metric Spaces

Definition 2.5. Assume that X be a set $(X \neq \phi)$ and $S : X^3 \to \mathbb{R}^+$ be a function that is satisfying the following conditions for all $\delta_1,\ \delta_2,\ \delta_3,\ b \in X$.
1- $S(\delta_1,\ \delta_2,\ \delta_3) = 0$ if and only if $\delta_1 = \delta_2 = \delta_3$.
2- $S(\delta_1,\ \delta_2,\ \delta_3) \leqslant S(\delta_1,\ \delta_1,\ b) + S(\delta_2,\ \delta_2,\ b) + S(\delta_3,\ \delta_3,\ b)$
Then the pair (X, S) is said to be an s-metric space.

Definition 2.6. Let (X, S) be an s-Metric Space. For $\rho > 0$ and $a \in X$, we define the open ball $B_s(a, \rho)$, the closed ball $B_s[a, \rho]$ with center a and radius ρ as follows:
$B_s(a, \rho) = \{b \in X : S(b,\ b,\ ,a) < \rho\}$,
$B_s[a, \rho] = \{b \in X : S(b,\ b,\ ,a) \leqslant \rho\}$,

The topology induced by the s-metric space is the topology generated by the base of all open balls in X.

Definition 2.7. Supposing that (X, S) be an s-Metric Space and δ_n be a sequence in X. Then, we obtain followings.

i- If δ_n is converge to δ in X for a given $\epsilon > 0$, then there exist N in $\mathbb{N}$ such that $S(\delta_n, \delta_n, \delta) \leqslant \epsilon,\ \forall n > N$.
ii- If δ_n is a Cauchy sequence in X for a given $\epsilon > 0$, then there exist N in $\mathbb{N}$, such that
$S(\delta_n, \delta_n, \delta_m) \leqslant \epsilon,\ \forall n,\ m > N$.
iii- An s-Metric Space (X, S) is complete if every Cauchy sequence is convergent to $x \in X$.

Lemma 2.2. *Let us (X, S) be an s-Metric Space under the assumption. In this case, following inequality is satisfied*

$$S(\delta,\ \delta,\ \zeta) = S(\zeta,\ \zeta,\ \delta),\ \forall \delta,\ \zeta \in X.$$

Lemma 2.3. *Assuming that (X, S) be an s-metric space and $\delta_n,\ \zeta_n$ be sequences converge to δ and ζ in X, respectively. Then, $S(\delta_n,\ \delta_n,\ \zeta_n)$ converges to $S(\delta,\ \delta,\ \zeta)$ in $\mathbb{R}^+$.*

Example 2.2. *$\mathbb{R}$ represents the set af all real numbers. So,*

$$S(\delta_1,\ \delta_2,\ \delta_3) = |\delta_1 - \delta_3| + |\delta_2 - \delta_3|$$

is an s-metric on $\mathbb{R}$, for all $\delta_1,\ \delta_2,\ \delta_3 \in \mathbb{R}$. Besides, $(\mathbb{R},\ S)$ is called as the usual s-metric space.

3. Main Results

3.1. C^*-Algebra Valued s-Metric Space

Definition 3.1. Assume that X be a set $(X \neq \emptyset)$, K be a C^*- algebra and S be a function defined from $X \times X \times X$ up to K_+, that is satisfying following conditions
S1- $S(\delta_1,\ \delta_2,\ \delta_3) = 0_{K_+}$ if and only if $\delta_1 = \delta_2 = \delta_3$.
S2- $S(\delta_1,\ \delta_2,\ \delta_3) \leqslant S(\delta_1,\ \delta_1,\ b) + S(\delta_2,\ \delta_2,\ b) + S(\delta_3,\ \delta_3,\ b)$.
For all $\delta_1,\ \delta_2,\ \delta_3,\ b \in X$.
The space (X, K, S) is said to be a C^*- Algebra Valued s-Metric Space.

Definition 3.2. Let (X, K, S) be a C^*- Algebra Valued s-Metric Space with Given $\rho > 0$ and $a \in X$. Then the open ball $B_{s_k}(a, \rho)$ and the closed ball $B_s[a, \rho]$ with center a and radius ρ are defined as follows;

$$B_{s_k}(a, \rho) = \{b \in X : \|S(b,\ b,\ ,a)\|_{K_+} < \rho\},$$

$$B_{s_k}[a, \rho] = \{b \in X : \|S(b,\ b,\ ,a)\|_{K_+} \leqslant \rho\},$$

As we said in previous section, the topology induced by the C^*- algebra valued s-metric space is the topology generated by the base of all open balls in X.

Definition 3.3. Supposing that (X, K, S) be a C^*- Algebra valued s-Metric Space and $\{\delta_n\}$ be a sequence in X. Then, if δ_n is converge to δ in X with given $\epsilon > 0$, then there exist N in $\mathbb{N}$, such that

$$\|S(\delta_n, \delta_n, \delta)\|_{K_+} \leqslant \epsilon,\ \forall n > N.$$

Definition 3.4. Assuming that (X, K, S) be a C^*- algebra valued s-metric space and $\{\delta_n\}$ be a sequence in X. Then, if δ_n is a Cauchy sequence in X with given $\epsilon > 0$, then there exist N in $\mathbb{N}$, such that

$$\|S(\delta_n, \delta_n, \delta_m)\|_{K_+} \leqslant \epsilon, \ \forall n, \ m > N.$$

Definition 3.5. The tripled(X, K, S) C^*- algebra valued s-metric space is complete space if every Cauchy sequence is convergent to $x \in X$.

Lemma 3.1. *If (X, K, S) is a C^*- Algebra Valued s-Metric Space, then, we can say that*

$$S(\delta, \ \delta, \ \zeta) = S(\zeta, \ \zeta, \ \delta), \ \forall \delta, \ \zeta \in X.$$

is satisfied.

Lemma 3.2. *If (X, K, S) is a C^*- algebra valued s-metric space and also $\delta_n \to \delta$, and $\zeta_n \to \zeta$ in K_+, then*

$$S(\delta_n, \ \delta_n, \ \zeta_n) \to S(\delta, \ \delta, \ \zeta) \ in \ K_+.$$

Corollary 3.1. *Let X_1 and X_2 be C^*- algebra valued s-metric spaces and $h : X_1 \to X_2$. So, h is continuous function at $\delta \in X_1$ if and only if $h(\delta_n) \to h(\delta)$ whenever $\delta_n \to \delta$.*

Example 3.1. *Let $X = T$ (unit circle) and $K = \mathbb{C}^2$ with*

$$\|(\mu_1, \mu_2)\| = Max\{|\mu_1|, |\mu_2|\}.$$

If we put order on K as follows

$$(\mu_1, \mu_2) \leqslant (\eta_1, \eta_2) \ if \ and \ only \ if \ |\mu_1| \leqslant |\eta_1|, \ and \ |\mu_2| \leqslant |\eta_2|,$$

then, we get that $(K_+, \leqslant)$ is partially order relation. (Since $K_+ = K = \mathbb{C}^2$.)
Besides, if $S : X \times X \times X \to K_+$ is defined as

$$S(x_1, \ x_2, \ x_3) = (Max\{|x_1 - x_3|, |x_2 - x_3|\}, 0),$$

then, (X, K, S) is a C^*- algebra valued s-metric space.

3.2. Main Theorems

Now, we present an implicit relation to discuss some fixed point theorems on a C^*- algebra valued s-metric space. Let M_k' be the family of all continuous operators, and $M_k \in M_k'$ such that $M_k : K_+^5 \to K_+$. Then, we can consider the following conditions:
(C1) Consider that $k_1, \ k_2, \ k_3 \in K_+$. If $k_2 \leqslant M_k(k_1, \ k_1, \ 0, \ k_3, \ k_2)$ and $k_3 \leqslant 2k_1 + k_2$, then $k_2 \leqslant k \ k_1 \ k^*$, for some $k \in K_+$, $\|k\| < 1$.

(C2) If $k_2 \leqslant M_k(k_2, \ 0, \ k_2, \ k_2, \ 0)$, then $k_2 = 0_k$, for all $k_2 \in K$.

Theorem 3.1. *Let $T_k : (X, \ K, \ S) \to (X, \ K, \ S)$ be a map mapping, where $(X, \ K, \ S)$ is a complete C^*- algebra valued s-metric space, and*

$$S(T_k x_1, T_k x_2, T_k x_3) \leqslant M_k(S(x_1, x_1, x_2), S(T_k x_1, T_k x_1, x_1), S(T_k x_1, T_k x_1, x_2),$$

$$S(T_kx_2,T_kx_2,x_1),S(T_kx_2,T_kx_2,x_2)) \tag{3.1}$$

for all $x_1,\ x_2,\ x_3 \in X$ and some $M_k \in M'_k$. Then we have the following results:
(1) If M_k satisfies the condition (C1), then T_k has a fixed point.
(2) If M_k satisfies the condition (C2) and T_k has a fixed point, then the fixed point is unique.

Proof. (1) For each $n \in \mathbb{N}$, assume that $\delta_{n+1} = T_k(\delta_n)$.
Using (3.1) and Lemma 3.6, we get

$$\begin{aligned}
S(\delta_{n+1},\delta_{n+1},\delta_{n+2}) &= S(T_k(\delta_n),T_k(\delta_n),T_k(\delta_{n+1}))\\
&\leqslant M_k(S(\delta_n,\delta_n,\delta_{n+1}),S(\delta_{n+1},\delta_{n+1},\delta_n),S(\delta_{n+1},\delta_{n+1},\delta_{n+1}),S(\delta_{n+2},\delta_{n+2},\delta_n),S(\delta_{n+2},\delta_{x+2},\delta_{n+1}))\\
&= M_k(S(\delta_n,\delta_n,\delta_{n+1}),S(\delta_n,\delta_n,\delta_{n+1}),0,S(\delta_n,\delta_n,\delta_{n+2}),S(\delta_{n+1},\delta_{n+1},\delta_{n+2}))
\end{aligned}$$

By (S2) and Lemma 3.6, we have

$$\begin{aligned}
S(\delta_{n+1},\delta_{n+1},\delta_{n+2}) &\leqslant 2S(\delta_n,\delta_n,\delta_{n+1})+S(\delta_{n+2},\delta_{n+2},\delta_{n+1})\\
&= 2S(\delta_n,\delta_n,\delta_{n+1})+S(\delta_{n+1},\delta_{n+1},\delta_{n+2})
\end{aligned}$$

Since M_k is satisfying the condition (C1), there exist a $k \in K_+$ with $\|k\| \leqslant 1$, such that

$$\begin{aligned}
S(\delta_{n+1},\delta_{n+1},\delta_{n+2}) &\leqslant kS(\delta_n,\delta_n,\delta_{n+1})k^*\\
&\leqslant k^{n+1}S(\delta_0,\delta_0,\delta_1)(k^*)^{n+1} \qquad (3.2)
\end{aligned}$$

Considering (S2), Lemma 3.6 and Lemma 3.2, we obtain

$$\begin{aligned}
S(\delta_n,\delta_n,\delta_m) &\leqslant 2S(\delta_n,\delta_n,\delta_{n+1})+S(\delta_m,\delta_m,\delta_{n+1})\\
&\leqslant 2S(\delta_n,\delta_n,\delta_{n+1})+S(\delta_{n+1},\delta_{n+1},\delta_m)\\
&\leqslant ...\\
&\leqslant 2k^nS(\delta_0,\delta_0,\delta_1)(k^*)^n+...+2k^{m-1}S(\delta_0,\delta_0,\delta_1)(k^*)^{m-1}\\
&\leqslant 2(k^nB^{\frac{1}{2}})(B^{\frac{1}{2}}k^n)^*+...+2(k^{m-1}B^{\frac{1}{2}})(B^{\frac{1}{2}}k^{m-1})^*,\ \textit{where } B=S(\delta_0,\delta_0,\delta_1)\\
&= \sum_{r=n}^{m-1} 2|k^rB^{\frac{1}{2}}|^2,
\end{aligned}$$

for all $n < m$.

Hence, we have

$$\begin{aligned}
\|S(\delta_n, \delta_n, \delta_m)\|_{K_+} &\leqslant \|\sum_{r=n}^{m-1} 2|k^r B^{\frac{1}{2}}|^2\| \\
&\leqslant \sum_{r=n}^{m-1} \|k\|^{2r} \|B^{\frac{1}{2}}\|^2 \\
&= \|B^{\frac{1}{2}}\|^2 \sum_{r=n}^{m-1} \|k^r\|^2 \\
&= \|B^{\frac{1}{2}}\|^2 \frac{2\|k\|^{2n}}{1-\|k\|}
\end{aligned}$$

by Geometry series.
But, we also know that

$$\lim_{n\to\infty} \|B^{\frac{1}{2}}\|^2 \frac{2\|k\|^{2n}}{1-\|k\|} = 0_B$$

since $\|k\| < 1$.
Therefore, δ_n is a Cauchy sequence in the C^*- algebra valued s-metric space (X, K, S).
Also (X, K, S) is complete, thus;

$$\lim_{n\to\infty} \delta_n = \delta.$$

Using inequality (3.1), we get

$$\begin{aligned}
S(\delta_{n+1}, \delta_{n+1}, T_k\delta) &= S(T_k(\delta_n), T_k(\delta_n), T_k\delta) \\
&\leqslant M_k(S(\delta_n, \delta_n, \delta), S(T_k\delta_n, T_k\delta_n, \delta), S(T_k\delta_n, T_k\delta_n, \delta_n), S(T_k\delta, T_k\delta, \delta_n), S(T_k\delta, T_k\delta, \delta)) \\
&= M_k(S(\delta_n, \delta_n, \delta), S(\delta_{n+1}, \delta_{n+1}, \delta), S(\delta_{n+1}, \delta_{n+1}, \delta_n), S(T_k\delta, T_k\delta, \delta_n), S(T_k\delta, T_k\delta, \delta))
\end{aligned}$$

If we take take limit for $n \to \infty$ and using Lemma 3.7, we obtain

$$S(\delta, \delta, T_k\delta) \leqslant M(0, 0, 0, S(T_k\delta, T_k\delta, \delta), S(T_k\delta, T_k\delta, \delta)).$$

Thus; $S(\delta, \delta, T_k\delta) \leqslant k.0.k^*$ since M_k satisfies (C1).
So, this implies that $T_k\delta = \delta$ and δ is a fixed point of T_k.

(2) Let δ_1, δ_2 be fixed points of T_k. From (3.1) and Lemma 3.6, we have

$$\begin{aligned}
S(\delta_1, \delta_1, \delta_2) &= S(T_k\delta_1, T_k\delta_1, T_k\delta_2) \\
&\leqslant M_k(S(\delta_1, \delta_2, \delta_2), S(T_k\delta_1, T_k\delta_1, \delta_1), S(T_k\delta_1, T_k\delta, \delta_2), S(T_k\delta_2, T_k\delta_2, \delta_1), S(T_k\delta_2, T_k\delta_2, \delta_2)) \\
&= M_k(S(\delta_1, \delta_1, \delta_2), 0, S(\delta_1, \delta_1, \delta_2), S(\delta_2, \delta_2, \delta_1), 0) \\
&= M_k(S(\delta_1, \delta_1, \delta_2), 0, S(\delta_1, \delta_1, \delta_2), S(\delta_1, \delta_1, \delta_2), 0)
\end{aligned}$$

Hence $S(\delta_1, \delta_1, \delta_2) = 0$ since M_k is satisfying (C2).So, it is trivial that $\delta_1 = \delta_2$. $\square$

Example 3.2. *Let $X = T$ (unit circle) and $K = \mathbb{C}^3$ with $\|(\mu_1, \mu_2, \mu_3)\| = Max\ \{|\mu_1|, |\mu_2|, |\mu_3|\}$. If we put order on K as follows*

$$(\mu_1, \mu_2, \mu_3) \leqslant (\eta_1, \eta_2, \eta_3)\ if\ and\ only\ if\ |\mu_1| \leqslant |\eta_1|,\ |\mu_2| \leqslant |\eta_2|,\ and\ |\mu_3| \leqslant |\eta_3|$$

then, we obtain that $(K_+, \leqslant)$ is partially order relation. (Since $K_+ = K = \mathbb{C}^3$).

Besides, suppose that $S : X \times X \times X \to K_+$ is defined as

$$S(\delta_1, \delta_2, \delta_3) = (Max\ \{|\delta_1 - \delta_3|, |\delta_2 - \delta_3|\}, 0, 0).$$

and $M_k(k_1, k_2, k_3, k_4, k_5) = \frac{1}{4}\ k_1 = \frac{1}{2}\ k_1\ \frac{1}{2}$.

For $T_k(\delta) = \begin{cases} \frac{1}{2} & : \|\delta\| < 1 \\ \frac{1}{4} & : \|\delta\| = 1 \end{cases}$.

Then T_k has unique fixed point on $\delta = \frac{1}{2}$, Since conditions (C1) and (C2) in theorem 3.2 are satisfying.

Corollary 3.2. *Assume that T_k is a self map on the complete C^*- algebra valued s-metric space $(X,\ K,\ S)$ and*

$$S(T_k\delta, T_k\delta, T_k\zeta) \leqslant R\ S(\delta, \delta, \zeta)R^*$$

for some $R \in K_+$, $\|R\| < 1$ while $\delta, \zeta \in X$. Then, T_k has a unique fixed point in X.

Proof. We can easily see that the proof is trivial by using theorem 3.10 and

$$M_k(k_1, k_2, k_3, k_4, k_5) = Rk_1R^*.$$

For $R \in K_+$, $\|R\| < 1$ and $k_1, k_2, k_3, k_4, k_5 \in K_+$. □

Remark 3.1. *Also, we should note if $(K, \leqslant)$ is a partially order relation, then maximum of two element in K is the supremum.*

Corollary 3.3. *If T_k is a self map on the complete C^*- algebra valued s-metric space $(X,\ K,\ S)$ and*

$$S(T_k\delta, T_k\delta, T_k\zeta) \leqslant w\ Max\{S(T_k\delta, T_k\delta, \delta),\ S(T_k\zeta, T_k\zeta, \zeta)\}w^*$$

for some $w \in K_+$, $\|w\| < 1$ and for all $\delta, \zeta \in X$, then T_k has a unique fixed point in X.

Proof. In a similar way of the proof of corollary 3.2. we can use theorem 3.10 and

$$M_k(k_1, k_2, k_3, k_4, k_5) = w\ max\{k_2, k_5\}\ w^*,$$

for some $w \in K_+$, $\|w\| < 1$ and $k_1, k_2, k_3, k_4, k_5 \in K_+$, to prove corollary 3.3. □

Example 3.3. *Let $X = [0, 1]$ and $K = M_2(C)$ with $\|D\| = Max_i|d_i|$. If we put order on K as follows*

$$D \leqslant F\ if\ and\ only\ if\ |d_i| \leqslant |f_i|,\ for\ all\ i$$

then, we obtain that $(K_+, \leqslant)$ *is a partially order relation. (Since* $K_+ = K = M_2(C)$*).*
Also, if $S : X \times X \times X \to K_+$ *is defined as*

$$S(\delta_1, \delta_2, \delta_3) = \begin{pmatrix} Max\{\delta_1, \delta_2, \delta_3\} & 0 \\ 0 & 0 \end{pmatrix}.$$

and

$$T_k(\delta) = \frac{\delta}{9},$$

then T_k *has a unique fixed point on* $\delta = 0$. *It follows from Corollary 3.13, we have*

$$S(T_k\delta, T_k\delta, T_k\zeta) = \begin{pmatrix} Max\{\frac{\delta}{9}, \frac{\zeta}{9}\} & 0 \\ 0 & 0 \end{pmatrix} \leqslant \frac{1}{3} \begin{pmatrix} Max\{\delta, \zeta\} & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{3}.$$

But $max\{\delta, \zeta\} = Max\{S(T_k\delta, T_k\delta, \delta),\ S(T_k\zeta, T_k\zeta, \zeta)\}$. *Thus;*

$$S(T_k\delta, T_k\delta, T_k\zeta) \leqslant \frac{1}{3} \begin{pmatrix} Max\{S(T_k\delta, T_k\delta, \delta),\ S(T_k\zeta, T_k\zeta, \zeta)\} & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{3}.$$

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