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An efficient and accurate Sine pseudo-spectral method for the nonlinear Schrödinger equation with wave operator

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Abstract: In this study, an efficient and accurate Sine pseudo-spectral method is constructed for solving the nonlinear Schrödinger equation with wave operator (NLSW). In this method, a modified leap-frog finite difference method is adopted for time discretization and a Sine pseudo-spectral method is employed for spatial discretization. The proposed method is proved to preserve the total energy in the discrete sense. We report several numerical results to show that, without any restriction on the grid ratio, the proposed method is of spectral accuracy in space and second-order accuracy in time. The energy conservation law of the proposed method is also verified numerically.

Key words: Nonlinear Schrödinger equation with wave operator; SPS method; Energy conservation; High accuracy

1. Introduction

In this paper, we consider the nonlinear Schrödinger equation with wave operator (NLSW) in two dimensions

$$\partial_{tt}u - \Delta u + i\partial_t u + f(|u|^2)u = 0, \quad (x,y) \in \mathbb{R}^2,\ t > 0, \tag{1.1}$$

with (l_1, l_2)-periodic boundary conditions

$$u(x+l_1, y, t) = u(x,y,t), \quad u(x, y+l_2, t) = u(x,y,t), \quad (x,y) \in \mathbb{R}^2,\ t > 0, \tag{1.2}$$

and initial conditions

$$u(x,y,0) = \varphi(x,y), \quad \partial_t u(x,y,0) = \psi(x,y), \quad (x,y) \in \mathbb{R}^2, \tag{1.3}$$

where $i = \sqrt{-1}$ is the complex unit, $u = u(x,y,t)$ is the unknown complex-valued function, x, y is the spatial variables, t is the time, Δ is the two-dimensional Laplace operator, f, ϕ, ψ is three given functions .

NLSW is one of most important nonlinear Schrödinger-type equations, it is widely used to describe many physical phenomena. In fact, the solution of the initial-periodic boundary value problem (1.1)-(1.3) satisfies the following energy conservation law:

$$E(t) := \int_\Omega (|\partial_t u|^2 + |\nabla u|^2 + F(|u|^2))dxdy \equiv E(0), \quad t \geq 0, \tag{1.4}$$

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where $F(\rho) = \int_0^{\rho} f(s)ds$, and $\Omega = [0, l_1] \times [0, l_2]$.Hence, to design an energy-preserving and unconditionally stable numerical scheme for solving NLSW is an interesting and important issue.

In the last several decades, various numerical methods have been developed in the literatures for solving NLSW, including the finite difference method, the finite element method and the spectral method. Among them, finite difference method is particularly popular[1–6]. Conservative finite difference schemes for NLSW in one dimension are proposed in [6, 7], based on an a priori maximum estimate of the numerical solution, the optimal error estimates are established. In [1], Bao and Cai presented a conservative but nonlinear Crank-Nicolson finite difference scheme and a semi-implicit finite difference scheme for solving the NLSW with high oscillation in time, where the temporal oscillatory nature is described by a small positive parameter ε, they established the uniform l^2 and semi-H^1 error estimates in a ingenious and rigorous way. Wang analysised a semi-implicit compact finite difference (SICFD) method, and proved the uniform l_∞ norm error bounds in ε to be of $O(h^4 + \tau)$ and $O(h^4 + \tau^{2/3})$ with time step τ and mesh size h for well-prepared and ill-prepared initial date in [8].

In the study of finite element methods (FEMs), Galerkin FEMs were widely used. The optimal error estimates are built in the sense by using the error-splitting technique along with the standard energy method [9–11], where the choice of the time step size is independent of the spatial mesh size. Cai, He and Pan proposed a conservative finite element method for solving the cubic NLSW, and established the optimal error estimate in L^2 norm without any restriction on the grid ratio In [12].

In the field of spectral method, the Fourier pseudo-spectral method and the Sine pseudo-spectral method are widely used method to solve differential equations, it converges exponentially fast in space if the exact solution is smooth, i.e., it has spectral accuracy in space. In [13], Ji and Zhang proposed an exponential wave integrator Fourier pseudo-spectral method in one dimension, they proved that the method has spectral accuracy in space and second-order accuracy in time in H^1 norm. In [14], Bao and Cai proposed an exponential wave integrator Sine pseudo-spectral (EWI-SP) method for NLSW with oscillatory characteristics in time, and they analyzed that the EWI-SP method is of the uniform spectral accuracy in space and second order accuracy in time for well-prepared initial data.Nevertheless, there are some restrictions on the grid ratios for two- and three-dimensional cases.

In this work, we introduce an efficient Sine pseudo-spectral method for solving the nonlinear Schrödinger equation with wave operator (1.1)-(1.3), and prove the conservation of the scheme.

The rest paper is organized as follows. In Section 2, we propose a SPS method for solving the NLSW, and prove the conservation properties of SPS method. In Section 3, numerical results are reported to verified effectiveness of the method and to confirm the conservation law.

2. Numerical methods and energy conservation

For a positive integer N, choose time-step $\tau = T/N$ and denote the set of temporal steps to be $\Omega_\tau := \{t_n | t_n = n\tau, n = 0, 1, 2, ..., N\}$, where $0 < T < T_{max}$ with T_{max} the maximal existing time of the solution; choose mesh size $h_1 = l_1/M_1, h_2 = l_2/M_2$ with two positive integers M_1 and M_2, denote $h = \max\{h_1, h_2\}$, $h_{min} = \min\{h_1, h_2\}$, the set of spatial points to be $\Omega_h := \{(x_j, y_k) | (x_j, y_k) = (jh_1, kh_2), j = 0, 1, 2, ..., M_1 - 1, k = 0, 1, 2, ..., M_2 - 1\}$, and denote the set of the grid points in space and time to be $\Omega_h^\tau := \Omega_h \times \Omega_\tau$.

Given a grid function $w = \{w_{j,k}^n | (x_j, y_k, t_n) \in \Omega_h^\tau\}$, we denote

$$\delta_x^2 w_{j,k}^n = \frac{1}{h_1^2}(w_{j-1,k}^n - 2w_{j,k}^n + w_{j+1,k}^n), \quad \delta_y^2 w_{j,k}^n = \frac{1}{h_2^2}(w_{j,k-1}^n - 2w_{j,k}^n + w_{j,k+1}^n), \quad w_{j,k}^{\bar{n}} = \frac{1}{2}(w_{j,k}^{n+1} + w_{j,k}^{n-1})$$

$$\triangle_h w_{j,k}^n = \delta_x^2 w_{j,k}^n + \delta_y^2 w_{j,k}^n, \quad \delta_t^+ w_{j,k}^n = \frac{1}{\tau}(w_{j,k}^{n+1} - w_{j,k}^n), \quad \delta_t w_{j,k}^n = \frac{1}{2\tau}(w_{j,k}^{n+1} - w_{j,k}^{n-1}),$$

Let $\mathbb{V} = \{u | u = \{u_{j,k} | (x_j, y_k) \in \Omega_h\}$ and u is periodic $\}$ be the space of periodic gird function on Ω_h. For any two grid functions $u, v \in \mathbb{V}_h$, define the discrete inner product as

$$(u, v)_h = h_1 h_2 \sum_{j=0}^{M_1-1} \sum_{k=0}^{M_2-1} u_{j,k} \overline{v}_{j,k},$$

where $\overline{v}$ is the complete conjugate of v.

For any grid function $u \in \mathbb{V}_h$, the discrete L^2-norms of u and its difference quotients, and the discrete $L^\infty-$norms of u are defined, respectively, as

$$||u||_h = \sqrt{(u, u)_h}, \quad ||\delta_x^+ u||_h = \sqrt{(\delta_x^+ u, \delta_x^+ u)_h}, \quad ||\delta_y^+ u||_h = \sqrt{(\delta_y^+ u, \delta_y^+ u)_h},$$

$$|u|_{h,1} = \sqrt{||\delta_x^+ u||_h^2 + ||\delta_y^+ u||_h^2}, \quad |u|_{h,2} = \sqrt{(\Delta_h u, \Delta_h u)_h}, \quad ||u||_{h,1} = \sqrt{||u||_h^2 + |u|_{h,1}^2},$$

$$||u||_{h,2} = \sqrt{||u||_h^2 + |u|_{h,1}^2 + |u|_{h,2}^2}, \quad ||u||_\infty = \max_{(x_j, y_k) \in \Omega_h} |u_{j,k}|.$$

Here it should be pointed out that the discrete norms $||\delta_x^+ u||_h, ||\delta_y^+ u||_h, |u|_{h,1}, |u|_{h,2}$ defined above are mere semi-norms of the grid function u.

For the sake of Simplicity, we use $u_{j,k}^n$ and $U_{j,k}^n$ to denote the exact value and the approximation of $u(x_j, y_k, t_n)$, respectively. We use C to represents a general normal number independent of all discretized parameters, which may have different values under different conditions. The mesh sizes h_1, h_2 are chosen to satisfy $h \leq C h_{min}$.

2.1. SPS approximation of spatial derivatives

Introduce an interpolation space $S_M^{''}$ as

$$S_M^{''} := \text{span}\{g_p(x) g_q(y), p = 0, 1, 2, ..., M_1 - 1, q = 0, 1, 2, ..., M_2 - 1\}$$

with $g_p(x)$ and $g_q(y)$ are trigonometric polynomial given respectively by

$$g_p(x) = \frac{2}{M_1} \sum_{l=1}^{M_1-1} sin(l\mu_1 x_p) sin(l\mu_1 x), \quad g_q(y) = \frac{2}{M_2} \sum_{l=1}^{M_2-1} sin(m\mu_2 y_q) sin(m\mu_2 y), \tag{2.1}$$

where $\mu_1 = \frac{\pi}{l_1}$, $\mu_2 = \frac{\pi}{l_2}$.

Define the interpolation operator $I_M : L^2(\Omega) \to S_M^{''}$ as

$$I_M u(x, y) = \sum_{j=1}^{M_1-1} \sum_{k=1}^{M_2-1} u(x_j, y_k) g_j(x) g_k(y), \tag{2.2}$$

where $g_j(x_l) = \delta_j^l, g_k(x_m) = \delta_k^m$.

For a function $u \in L^2(\Omega)$, to obtain the derivative $\partial_x^{\alpha_1}\partial_y^{\alpha_2} I_M u(x,y)$, we differentiate (2.2) and evaluate the resulting expression at the points (x_j, y_k),

$$\partial_x^{s_1}\partial_y^{s_2} I_M u(x_j,y_k) = \sum_{l=1}^{M_1-1}\sum_{m=1}^{M_2-1} u(x_l,y_m)\frac{d^{s_1}g_l}{dx^{s_1}}(x_j)\frac{d^{s_2}g_m}{dy^{s_2}}(y_k) = (D_{s_1}^x \mathbf{u}(D_{s_2}^y)^\top)_{j,k},$$

where $D_{s_1}^x$ is an $(M_1-1)\times(M_1-1)$ matrix and $D_{s_2}^y$ is an $(M_2-1)\times(M_2-1)$ matrix, respectively. With elements given by

$$(D_{s_1}^x)_{j,l} = \frac{d^{s_1}g_l}{dx^{s_1}}(x_j) = -\frac{2}{M_1}\sum_{p=1}^{M_1-1}(l\mu_1)^2 sin(p\mu_1 x_l)sin(l\mu_1 x_j),$$

$$(D_{s_2}^y)_{k,m} = \frac{d^{s_2}g_l}{dy^{s_2}}(y_k) = -\frac{2}{M_2}\sum_{p=1}^{M_2-1}(l\mu_2)^2 sin(p\mu_2 y_m)sin(l\mu_2 y_k),$$

and $\mathbf{u} = u(x_j, y_k)$ is an $(M_1-1)\times(M_2-1)$ matrix. Note that D_s^x and D_s^y are real symmetric/antisymmetric matrices when s is even/odd. For second derivatives, we have

$$\partial_{xx} I_M u(x_j,y_k) = (D_2^x \mathbf{u})_{j,k}, \quad \partial_{yy} I_M u(x_j,y_k) = (\mathbf{u} D_2^y)_{j,k},$$

where the symmetry of D_2^y is used.

2.2. ESPS methed

By using the Sine pseudo-spectral method and the finite difference method to discrete the NLSW equation (1.1)-(1.3) in space and in time, respectively, we obtain the following full-discrete numerical method

$$\delta_t^2 U_{j,k}^n - (D_2^x U^{\bar{n}})_{j,k} - (U^{\bar{n}} D_2^y)_{j,k} + i\delta_t U_{j,k}^n + f(|U_{j,k}^n|^2)U_{j,k}^n = 0, \tag{2.3}$$

for $n = 1, 2, ..., N-1$. In order to start the above three-level method, we give the numerical solution at the first time step by Taylor's expansion,

$$U_{j,k}^1 = \varphi(x_j,y_k) + \tau\psi(x_j,y_k) + \frac{\tau^2}{2}\phi(x_j,y_k), \tag{2.4}$$

where

$$\phi(x,y) = \Delta\varphi(x,y) - i\psi(x,y) - f(|\varphi(x,y)|^2)\varphi(x,y).$$

For convenience, the SPS (2.3) can be written in a matrix form

$$\delta_t^2 U^n - D_2^x U^{\bar{n}} - U^{\bar{n}} D_2^y + i\delta_t U^n + f(|U^n|^2)U^n = 0, \tag{2.5}$$

for $n = 1, 2, ..., N-1$, where $U^n = (U_{j,k}^n) \in \mathbb{V}_h$.

Here, we introduce two semi-norms over $\mathbb{V}_h$ as follows

$$|v|_1 := \sqrt{(-D_2^x v, v)_h + (-vD_2^y, v)_h}, \quad v \in \mathbb{V}_h, \tag{2.6}$$

$$|v|_2 := \sqrt{(D_2^x v + v D_2^y, D_2^x v + v D_2^y)_h}, \quad v \in \mathbb{V}_h. \tag{2.7}$$

Define $M_1 \times M_1$ matrix A_2^x and $M_2 \times M_2$ matrix A_2^y as follows

$$A_2^x = -\frac{1}{h_1^2}\begin{pmatrix} 2 & -1 & 0 & ... & -1 \\ -1 & 2 & -1 & ... & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & ... & -1 & 2 & -1 \\ -1 & ... & 0 & -1 & 2 \end{pmatrix}, \; A_2^y = -\frac{1}{h_2^2}\begin{pmatrix} 2 & -1 & 0 & ... & -1 \\ -1 & 2 & -1 & ... & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & ... & -1 & 2 & -1 \\ -1 & ... & 0 & -1 & 2 \end{pmatrix}, \tag{2.8}$$

by using summation-by-parts formula and the definition of the semi-norms $|\cdot|_{h,1}$ and $|\cdot|_{h,2}$, we can see that

$$|v|_{h,1}^2 := (-A_2^x v, v)_h + (-vA_2^y, v)_h, \quad |v|_{h,2}^2 := (A_2^x v + vA_2^y, A_2^x v + vA_2^y)_h, \quad v \in \mathbb{V}_h. \tag{2.9}$$

There is a lemma to prove that $|v|_1$ and $|v|_2$ are equivalent to $|v|_{h,1}$ and $|v|_{h,2}$, before that, we need the following two lemmas.

Lemma 2.1. *[15] For matrices $A_2^x, A_2^y, D_2^x, D_2^y$, there exist relations*

$$A_2^x = F_{M_1}^* \Lambda_2^x F_{M_1}, \quad A_2^y = F_{M_2}^* \Lambda_2^y F_{M_2}, \quad D_2^x = F_{M_1}^* \tilde{\Lambda}_2^x F_{M_1}, \quad D_2^y = F_{M_2}^* \tilde{\Lambda}_2^y F_{M_2} \tag{2.10}$$

where F_M is the discrete Fourier transform with elements $(F_M)_{j,k} = \frac{1}{\sqrt{M}} e^{-i\frac{2\pi}{M}jk}$, F_M^* is the conjugate transpose matrix of F_M, and

$$\Lambda_2^x = diag(\lambda_{A_2^x,0}, \lambda_{A_2^x,1}, ..., \lambda_{A_2^x,M_1-1}), \quad \lambda_{A_2^x,j} = -\frac{4}{h_1^2} sin^2 \frac{j\pi}{M_1},$$

$$\Lambda_2^y = diag(\lambda_{A_2^y,0}, \lambda_{A_2^y,1}, ..., \lambda_{A_2^y,M_2-1}), \quad \lambda_{A_2^y,j} = -\frac{4}{h_2^2} sin^2 \frac{k\pi}{M_2},$$

$$\tilde{\Lambda}_2^x = diag(\lambda_{D_2^x,0}, \lambda_{D_2^x,1}, ..., \lambda_{D_2^x,M_1-1}), \quad \lambda_{D_2^x,j} = \begin{cases} -\mu_j^2 & if \quad 0 \le j \le M_1/2, \\ -\mu_{j-M_1}^2 & if \quad M_1/2 \le j \le M_1, \end{cases}$$

$$\tilde{\Lambda}_2^y = diag(\lambda_{D_2^y,0}, \lambda_{D_2^y,1}, ..., \lambda_{D_2^y,M_1-1}), \quad \lambda_{D_2^y,k} = \begin{cases} -\nu_k^2 & if \quad 0 \le k \le M_2/2, \\ -\nu_{k-M_2}^2 & if \quad M_2/2 \le k \le M_2. \end{cases}$$

Moreover, there are the following inequalities

$$0 \le \lambda_{A_2^x,j} \le \lambda_{D_2^x,j} \le -\frac{\pi^2}{4}\lambda_{A_2^x,j}, \quad j = 0, 1, ..., M_1 - 1,$$

$$0 \le \lambda_{A_2^y,k} \le \lambda_{D_2^y,k} \le -\frac{\pi^2}{4}\lambda_{A_2^y,k}, \quad k = 0, 1, ..., M_2 - 1.$$

Lemma 2.2. *For $A \in \mathbb{C}^{M_1 \times M_1}, B \in \mathbb{C}^{M_2 \times M_2}$ and $v, w \in \mathbb{V}_h$, there are the following identities*

$$(Av, w)_h = (v, A^* w)_h, \quad (vB, w)_h = (v, wB^*)_h.$$

Lemma 2.3. *For any grid function $v \in \mathbb{V}_h$, there are the following inequalities*

$$|v|_{h,1} \leq |v|_1 \leq \frac{\pi}{2}|v|_{h,1}, \tag{2.11}$$

$$|v|_{h,2} \leq |v|_2 \leq \frac{\pi^2}{4}|v|_{h,2}. \tag{2.12}$$

Theorem 2.1. *The ESPS scheme (2.3)-(2.4) preserves the total energy in the discrete sense, i.e.,*

$$E^n \equiv E^0, \quad n = 1, 2, \cdots, N-1, \tag{2.13}$$

where E^n is the called discrete energy defined as

$$E^n \triangleq ||\delta_t^+ U^n||_{L^2}^2 + \frac{1}{2}(|U^{n+1}|_1^2 + |U^n|_1^2) + \frac{1}{2}(F^{n+1} + F^n, 1)_h. \tag{2.14}$$

Proof. Computing the inner product of (2.3) with $U^{n+1} - U^{n-1}$ and taking the real part, we have

$$\begin{aligned}&\mathrm{Re}(\delta_t^2 U^n, U^{n+1} - U^{n-1})_h - \mathrm{Re}(\frac{1}{2}D_2^x(U^{n+1} + U^{n-1}) + \frac{1}{2}(U^{n+1} + U^{n-1})D_2^y,\\ &U^{n+1} - U^{n-1})_h + \mathrm{Re}(f(|U^n|^2)U^n, U^{n+1} - U^{n-1})_h = 0.\end{aligned} \tag{2.15}$$

We denote the m-th term of above equation by $I_m (m = 1, 2, 3)$, i.e.,

$$I_1 + I_2 + I_3 = 0. \tag{2.16}$$

Direct calculation gives

$$I_1 = ||\delta_t^+ U^n||_{L^2}^2 - ||\delta_t^+ U^{n-1}||_{L^2}^2 \tag{2.17}$$

$$I_2 = \frac{1}{2}(|U^{n+1}|_1^2 + |U^n|_1^2) - (|U^n|_1^2 + |U^{n-1}|_1^2) \tag{2.18}$$

$$\begin{aligned}I_3 =& h_1 h_2 \sum_{j=0}^{M_1-1} \sum_{k=0}^{M_2-1} \mathrm{Re}(f(|U_{j,k}^n|^2))U_{j,k}^n(\overline{U_{j,k}^{n+1}} - \overline{U_{j,k}^{n-1}}))\\ =& \frac{1}{2}(F^{n+1} + F^n, 1) - \frac{1}{2}(F^n + F^{n-1}, 1).\end{aligned} \tag{2.19}$$

Substituting (2.17)-(2.19) into (2.15) it is easy to get

$$||\delta_t^+ U^n||_{L^2}^2 - ||\delta_t^+ U^{n-1}||_{L^2}^2 + \frac{1}{2}(|U^{n+1}|_1^2 + |U^n|_1^2) - \frac{1}{2}(|U^n|_1^2 + |U^{n-1}|_1^2) + \frac{1}{2}(F^{n+1} + F^n, 1) - \frac{1}{2}(F^n + F^{n-1}, 1) = 0$$

where

$$\begin{aligned}&F_{j,k}^{n+1} = F_{j,k}^{n-1} + \mathrm{Re}(2f(|U_{j,k}^n|^2)U_{j,k}^n(\overline{U_{j,k}^{n+1}} - \overline{U_{j,k}^{n-1}})),\\ &j = 1, 2, ..., M_1 - 1, k = 1, 2, ..., M_2 - 1, n = 1, 2, ..., N-1,\end{aligned} \tag{2.20}$$

$$F_{j,k}^1 = F(|U_{j,k}^1|^2), \quad j = 1, 2, ..., M_1 - 1, k = 1, 2, ..., M_2 - 1, \tag{2.21}$$

$$F^0_{j,k} = F(|U^0_{j,k}|^2), \quad j = 1, 2, ..., M_1 - 1, k = 1, 2, ..., M_2 - 1, \tag{2.22}$$

This, together with the definition of E^n, immediately gives

$$E^n = E^{n-1}, \quad n = 1, 2, ..., N - 1.$$

This completes the the proof. □

3. Numerical results

In this section, we carry out several numerical examples to show the performance of the proposed ESPS method.

Example 3.1. *We consider the periodic-boundary and initial value problem of the NLS equation with wave operator*

$$\partial_{tt}u - \triangle u + i\partial_t u + |u|^4 u = g, \quad (x, y) \in \Omega, \ t > 0, \tag{3.1}$$

in two-dimensional space with $\Omega = [0, 2\pi] \times [0, 2\pi]$ *, where the function* $g = g(x, y, t)$ *is chosen correspondingly to the exact solution*

$$u(x, y, t) = \sin(x)\sin(y)\exp(-it). \tag{3.2}$$

Direct calculation gives that $g(x, y, t) = (2 + |u(x, y, t)|^4)u(x, y, t)$.

we test the convergence order in space and time directions, respectively. The convergence order is calculated by the following formula

$$\text{Order} = \frac{\ln(\text{error1}/\text{error2})}{\ln(\delta 1/\delta 2)} \tag{3.3}$$

where δs, error$s$ $(s = 1, 2)$ are step size and the corresponding error with step size δs , respectively. To test the convergence order in time direction, we choose sufficiently small grid size $(M_1 = M_2 = 64)$ to ignore the spatial error . Numerical results are given in s in Table 1. It is clear to see that a second-order convergence in time direction can be explicitly observed in the discrete H^1 norm. To test the convergence order in spatial direction, we choose sufficiently small time-step $\tau = 1.0e - 4$ to ignore the temporal error. It is shown that the new method is spectrally accurate in space Table 2. We take $g = 0$ at the example and present numerical results in Figure 1, to show the energy conservation of the ESPS.

Table 1. Time errors of ESPS at T=1.0 (Example 3.1)

τ	0.01	0.02	0.04	0.08
$\|\|e^N\|\|_{L^2}$	6.2864e-05	2.5014e-04	9.8973e-04	3.6737e-03
order	-	1.99	1.98	1.89
$\|\|e^N\|\|_{H^1}$	6.3009e-05	2.5072e-04	9.9201e-04	3.6822e-4
order	-	1.99	1.98	1.89

Table 2. Spatial errors of quadratic ESPS at T=1.0 (Example 3.1)

$M_1 \times M_2$	8×8	16×16	32×32	64×64
$\|\|e^N\|\|_{L^2}$	1.2346e-08	1.2346e-08	1.2312e-08	1.2346e-08
$\|\|e^N\|\|_{H^1}$	1.3095e-08	1.2678e-08	1.2415e-08	1.2375e-08

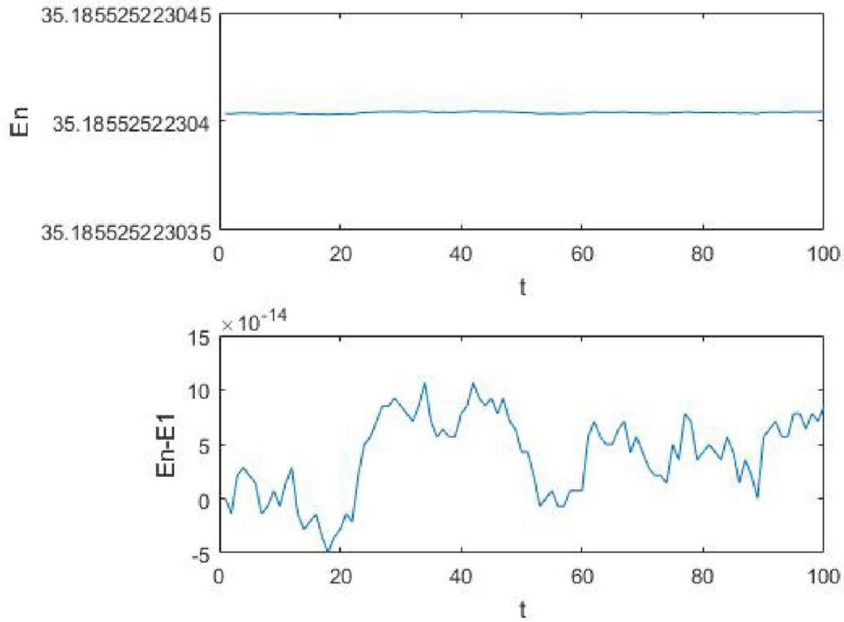

Figure 1. Total energy and its difference from the initial value computed with $g = 0, M_1 = M_2 = 16, \tau = 0.1$

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