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A variant of Drygas' functional equation with an endomorphism

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Abstract: We determine the solutions $f, h : S \to H$ of the following generalization of Drygas' functional equation

$$f(x+y) + f(x+\varphi(y)) = 2f(x) + h(y), \quad x, y \in S,$$

where S is a commutative semigroup, H is an abelian group which is uniquely 2-divisible and φ is an endomorphism of S (not necessarily involutive).

1. Set up, notation and terminology

Throughout the paper we work in the following framework and with the following notation and terminology. We use it without explicit mentioning.

S is a commutative semigroup [a set equipped with an associative composition rule $(x, y) \mapsto x + y$], the maps $\sigma, \tau : S \to S$ denotes an involutive automorphisms. That it is involutive means that $\sigma(\sigma(x)) = x, \tau(\tau(x)) = x$ for all $x \in S$, $\varphi : S \to S$ is an endomorphism and $(H, +)$ denotes an abelian group with neutral element 0 that is uniquely 2-divisible, i.e., for any $h \in H$ the equation $2x = h$ has exactly one solution $x \in H$.

A function $A : S \to H$ is said to be additive if $A(x + y) = A(x) + A(y)$ for all $x, y \in S$.

2. Introduction

Characterizing quasi-inner product spaces, Drygas considers in [8] the functional equation

$$f(x) + f(y) = f(x-y) + 2\left\{ f\left(\frac{x+y}{2}\right) - f\left(\frac{x-y}{2}\right) \right\}, \quad x, y \in \mathbb{R},$$

which can be reduced to the following equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \quad x, y \in \mathbb{R}. \tag{1}$$

This equation is known in the literature as Drygas' functional equation and is a common generalization of Jensen's equation $f(x+y) + f(x-y) = 2f(x)$, the quadratic equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$. The general solution of (1) was given by Ebanks et al. in [5]. It has the form

$$f(x) = A(x) + Q(x, x), \quad x \in \mathbb{R} \tag{2}$$

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where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a symmetric, bi-additive map (see also [16]). On an arbitrary group G, Drygas' equation takes the form

$$f(xy) + f(xy^{-1}) = 2f(x) + f(y) + f(y^{-1}), \quad x, y \in G. \tag{3}$$

Various authors studied the Drygas' functional equation, for example charifi et al. [1], Ebanks et al. [5], Fa$\breve{i}$ziev and Sahoo [17], Jung and Sahoo [16] , Łukasik [15], Szabo [7], Yang [6].

In [4], Fadli determined the solutions $f : S \to H$ of the following generalization of drygas' functional equation

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + f(y) + f(\sigma(y)), \quad x, y \in S, \tag{4}$$

where S is a commutative semigroup, H is a uniquely 2-divisible abelian group, and $\sigma : S \to S$ is an involutive automorphism. Replacing here σ by an arbitrary endomorphism φ of S, we obtain the following functional equation

$$f(x + y) + f(x + \varphi(y)) = 2f(x) + f(y) + f(\varphi(y)), \quad x, y \in S. \tag{5}$$

The aim of this article is first to find the solutions $f : S \to H$ of the functional equation (5) on abelian semigroups, and then to present an important consequence concerning the solutions $f, h : S \to H$ of the more general functional equation

$$f(x + y) + f(x + \varphi(y)) = 2f(x) + h(y), \quad x, y \in S, \tag{6}$$

that contains the two unknown functions f and h. This equation generalizes also Jensen's and the quadratic functional equations.

3. Solution of Drygas' functional equation with an endomorphism

Lemma 3.1 ([12], Lemma 3.2)**.** *Let T be a semigroup, and let $\phi : T \to T$ be an endomorphism. If $f : T \to H$ and $\Phi : T \times T \to H$ satisfy*

$$\Phi(x, y) = f(xy) + f(\phi(y)x), \quad x, y \in T,$$

then

$$2f(xyz) = \Phi(x, yz) - \Phi(\phi(z)x, y) + \Phi(xy, z) \quad \text{for all } x, y, z \in T.$$

Lemma 3.2. *If $f : S \to H$ is a solution of (5), then f satisfies Whitehead's functional equation*

$$f(x + y + z) = f(x + y) + f(x + z) + f(y + z) - f(x) - f(y) - f(z), \quad x, y, z \in S. \tag{7}$$

Proof. Let $x, y, z \in S$ be arbitrary. Using Lemma (3.1) with $T := S$ and $\Phi(x, y) := 2f(x) + f(y) + f(\varphi(y))$, we find that

$$\begin{aligned} 2f(x + y + z) &= [2f(x) + f(y + z) + f(\varphi(y) + \varphi(z))] - [2f(x + \varphi(z)) \\ &\quad + f(y) + f(\varphi(y))] + [2f(x + y) + f(z) + f(\varphi(z))] \\ &= 2f(x + y) + f(y + z) + f(\varphi(y) + \varphi(z)) - 2f(x + \varphi(z)) \\ &\quad + 2f(x) + f(z) + f(\varphi(z)) - f(y) - f(\varphi(y)). \end{aligned} \tag{8}$$

Now, by help of (5), we have

$$\begin{aligned} f(\varphi(y)+\varphi(z)) &= 2f(\varphi(y))+f(z)+f(\varphi(z))-f(\varphi(y)+z) \\ &= 2f(\varphi(y))+f(z)+f(\varphi(z))-[2f(z)+f(y) \\ &\quad +f(\varphi(y))-f(y+z)] \\ &= f(y+z)+f(\varphi(y))+f(\varphi(z))-f(y)-f(z), \end{aligned}$$

and

$$f(x+\varphi(z))=2f(x)+f(z)+f(\varphi(z))-f(x+z).$$

So, by replacing the last two equalities in (8) and dividing by 2, we get (7). □

The following result is an immediate consequence of [10, Theorem 6].

Theorem 3.1. *The general solution $f: S \to H$ of Whitehead's functional equation (7) is*

$$f(x)=A(x)+Q(x,x), \quad x \in S,$$

where $Q: S\times S \to H$ is a symmetric, bi-additive map and $A: S \to H$ is an additive function.

The following theorem solves the functional equation (5) on an arbitrary commutative semigroup.

Theorem 3.2. *The general solution $f: S \to H$ of the functional equation (5) is*

$$f(x)=Q(x,x)+A(x), \quad x \in S, \tag{9}$$

where $Q: S\times S \to H$ is a symmetric, bi-additive map such that $Q(x,\varphi(y))=-Q(x,y)$ for all $x,y\in S$, and where $A: S\to H$ is an additive function.

Proof. It is easy to check that any function f of the form (9) satisfies (5). Conversely, assume that f is a solution of (5). From Lemma 3.2, we see that f satisfies Whitehead's functional equation (7). According to Theorem 3.1, there exist a symmetric, bi-additive map $Q: S\times S\to H$ and an additive function $A: S\to H$ such that

$$f(x)=Q(x,x)+A(x), \quad x\in S.$$

By substituting f into (5), we find after some simplifications that $Q(x,\varphi(y))=-Q(x,y)$ for all $x,y\in S$. This completes the proof. □

4. Solution of a Pexider-Drygas' functional equation with an endomorphism

The following theorem solves the functional equation (6) on an arbitrary commutative semigroup.

Theorem 4.1. *The general solution $f,h: S\to H$ of the functional equation (6) is*

$$f(x)=Q(x,x)+A(x)+c$$

and

$$h(x)=2Q(x,x)+A(x)+A(\varphi(x)), \quad x\in S,$$

where $Q: S\times S\to H$ is a symmetric, bi-additive map such that $Q(x,\varphi(y))=-Q(x,y)$ for all $x,y\in S$, $A: S\to H$ is an additive function, and where $c\in H$ is a constant.

Proof. It is easy to check that any pair of functions of the form above satisfies (6). Conversely, assume that the pair (f,h) is a solution of (6). Replacing x with $\varphi(x)$ in (6), we get

$$f(\varphi(x)+y)+f(\varphi(x)+\varphi(y))=2f\circ\varphi(x)+h(y),\quad \text{for all } x,y\in S.$$

adding the last equation to the equation (6), we obtain

$$\begin{aligned} 2f(x)+2f\circ\varphi(x)+2h(y) &= f(x+y)+f(y+\varphi(x))+f(\varphi(y)+x)\\ &\quad +f(\varphi(y)+\varphi(x))\\ &= 2f(y)+2f\circ\varphi(y)+2h(x). \end{aligned}$$

Using the fact H is 2-torsion free, we get

$$h(x)-f(x)-f\circ\varphi(x)=h(y)-f(y)-f\circ\varphi(y),\quad \text{for all } x,y\in S.$$

From the last equation we infer that $h-f-f\circ\varphi$ is a constant in H, say c_1. Replace h in Eq. (6), we find

$$F(x+y)+F(x+\varphi(y))=2F(x)+F(y)+F(\varphi(y)),\quad x,y\in S,$$

with $F=f+\frac{c_1}{2}$, then F is solution of Eq. (5). From Theorem 3.2 we see that, there exist a symmetric, bi-additive map $Q:S\times S\to H$ satisfying $Q(\varphi(y),x)=-Q(x,y)$ for all $x,y\in S$, and an additive function $A:S\to H$ such that

$$F(x)=Q(x,x)+A(x),\quad x\in S.$$

Hence

$$f(x)=Q(x,x)+A(x)-\frac{c_1}{2},\quad x\in S.$$

Using $Q(\varphi(y),x)=-Q(x,y)$ for all $x,y\in S$, we obtain

$$h(x)=2Q(x,x)+A(x)+A\circ\varphi(x),\quad x\in S.$$

Then the pair (f,h) has the desired form with $c=-\frac{c_1}{2}$. □

As a consequence of Theorem 4.1 we have the following result.

Corollary 4.1. *The general solution $f:S\to H$ of the functional equation*

$$f(x+\tau(y))+f(x+\sigma(y))=2f(x)+2f(y),\quad x,y\in S, \tag{10}$$

is

$$f(x)=Q(x,x)+A(x)+c,\quad x\in S,$$

where $Q:S\times S\to H$ is a symmetric, bi-additive map such that $Q(x,\sigma(y))=-Q(x,\tau(y))$ and $Q(\sigma(x),\sigma(x))=Q(x,x)=Q(\tau(x),\tau(x))$ for all $x,y\in S$, $A:S\to H$ is an additive function such that $2A=A\circ\tau+A\circ\sigma$, and where $c\in H$ is a constant.

Proof. Replacing y with $\tau(y)$ in Eq. 12, we get

$$f(x+y)+f(x+\sigma\circ\tau(y))=2f(x)+2f\circ\tau(y),\quad x,y\in S. \tag{11}$$

From Theorem 4.1 we see that, there exist a symmetric, bi-additive map $Q : S\times S \to H$ such that $Q(x, \sigma(y)) = -Q(x, \tau(y))$ for all $x, y \in S$, and an additive function $A : S \to H$ such that

$$f(x) = Q(x, x) + A(x) + c$$

and

$$2f \circ \tau(x) = 2Q(x, x) + A(x) + A \circ \sigma \circ \tau(x), \quad x \in S.$$

Since Q is symmetric, we have $Q(\sigma(x), \sigma(y)) = -Q(\sigma(x), \tau(y)) = Q(\tau(x), \tau(y))$ for all $x, y \in S$. By substituting f into Eq. 12, we find after some simplifcations that $Q(\sigma(x), \sigma(x)) = Q(x, x) = Q(\tau(x), \tau(x))$ for all $x \in S$. This completes the proof. $\square$

Remark 4.1. *Theorem 4.1 is valid even if σ_1 and τ_1 are anti-automorphisms (not automorphisms) since, in this case, $\sigma_1 \circ \tau_1$ is an endomorphism.*

From the last remark we have the following result

Corollary 4.2. *Let $\sigma_1, \tau_1 \in Antihom(S, S)$ such that $\sigma_1 \circ \sigma_1 = \tau_1 \circ \tau_1 = id$ (where id denotes the identity map) The general solution $f : S \to H$ of the functional equation*

$$f(x + \tau_1(y)) + f(x + \sigma_1(y)) = 2f(x) + 2f(y), \quad x, y \in S, \tag{12}$$

is

$$f(x) = Q(x, x) + A(x) + c, \quad x \in S,$$

where $Q : S\times S \to H$ is a symmetric, bi-additive map such that $Q(x, \sigma_1(y)) = -Q(x, \tau_1(y))$, $Q(\sigma_1(x), \sigma_1(x)) = Q(x, x)$ and $Q(x, x) = Q(\tau_1(x), \tau_1(x))$ for all $x, y \in S$, $A : S \to H$ is an additive function such that $2A = A \circ \tau_1 + A \circ \sigma_1$, and where $c \in H$ is a constant.

Proof. the same proof as corollary 4.1 $\square$

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