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Galerkin finite element method for solving Newell-Whitehead-Segel equation

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Abstract: In this paper, a Galerkin finite element method is used to find the numerical solution of Newell-Whitehead-Segel equation, a nonlinear parabolic partial differential equation. Semidiscretization in space is obtained using finite elements in space and full discrete element equation is obtained using the Crank-Nicolson scheme. Newell-Whitehead-Segel equation is widely applied in mechanical and chemical engineering, ecology, biology and bio-engineering. Numerical results are interpreted graphically.

Key words: Reaction Diffusion Equation, Newell-Whitehead-Segel Equation, Galerkin Finite Element Method, Crank-Nicolson Scheme, Numerical Solution.

1. Introduction

Mathematical modeling plays a vital role in applied science. Nonequilibrium systems are shown in many extended states such as uniform, oscillatory, chaotic, and pattern states. Many stripe patterns, for instance, ripples in sand, stripes of seashells, occurs in a variety of spatially extended systems which can be described by a set of equation called amplitude equations. Newell-Whitehead-Segel (NWS) equation is one of the most important of amplitude equations, which describes the appearance of the stripe pattern in two dimensional systems [5, 9, 13]. Newell and Whitehead who developed the Newell-Whitehead-Segel equation while working on Benard's problem. NWS equation was applied to a number of problem to name a few, Rayleigh-Benard convection, Faraday instability, nonlinear optics, chemical reactions and biological systems.
Consider the Newell-Whitehead-Segel equation

$$\begin{aligned} &k\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} + lu - mu^q = 0, \quad in \quad \Omega_T, \\ &u(x,0) = f(x), \quad on \quad \Omega \times \{t=0\}, \\ &u(0,t) = u(1,t) = 0, \quad on \quad \partial\Omega \times [0,T] \end{aligned} \tag{1}$$

where l, m, k are real constants with $k>0$ and q is a positive integer. Assume $\Omega = (a,b)$ to be an open, bounded subset of $\mathbb{R}$ and $T>0$ to be the final time. Set $\Omega_T = \Omega \times (0,T)$. Here $f(x) : \Omega \to \mathbb{R}$ is the initial datum. The function $u(x,t)$ in the equation (1) is dependent function of spatial variable x, $(x \in R)$ and temporal variable t, $(t = 0)$. The term $\frac{\partial u}{\partial t}$ indicates the variation of function $u(x,t)$ with respect to time at specific location and the term $\frac{\partial^2 u}{\partial x^2}$ indicates the variation of function $u(x,t)$ with respect to spatial variable at

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some particular time. While the term $lu - mu^q$ describes the effect of source term. The function $u(x,t)$ in the Newell-Whitehead-Segel equation can be considered to represent the non-linear distribution of the temperature on an infinitely thin and long rod or can be considered as the velocity of a fluid in a pipe which is infinitely long with small diameter. Many researchers have been working to obtain the closest form of the exact solution for Newell-Whitehead-Segel equation. Numerical methods such as Adomain decomposition and multi-quadratic quasi-interpolation methods [5], Laplace Adomain Decomposition method [13], Differential transform method [1], Homotopy Pertubation method [9], and recently, New Iterative Method (NIM) proposed by Daftardar-Gejji and Jafari [13] were used to find the numerical solution of Newell-Whitehead-Segel equation. Galerkin finite method have been extensively used to find solutions of non linear differential equations. To mention a few from literature, it has been used for advection diffusion equation [10], for vibration of a one dimensional system with free end conditions [12], for two point boundary value problems [11, 14], for modified regularized long wave equation [8]. To the best of my knowledge, no researcher have so far made an attempt to find the numerical solution of the Newell-Whitehead-Segell equation using semi - discrete Galerkin finite element method and Crank Nicolson method. In this paper, we propose the numerical method for the governing equation, including semi discretization in space by Galerkin method and full discretization in space and time by Crank - Nicolson scheme. Crank-Nicolson method is the average of forward difference method and backward difference method. The numerical results are graphically interpreted using MatLab.

2. Numerical Methods

First let us linearize the Newell-Whitehead-Segel equation by considering its equilibrium solution. Let us consider q=2 in the equation (1). The linearization of equation (1) is done by considering the transformation

$$u(x,t) = u_0 + \epsilon v(x,t)$$

where u_0 is the equilibrium solution of the equation (1) with the boundary condition $u(0,t) = u(1,t) = 0$ and ϵ is small positive constant.

The linearized Newell-Whitehead-Segel equation is,

$$\begin{aligned} & k\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} + lv = 0 \quad && in \quad \Omega_T, \\ & v(x,0) = \frac{f(x)}{\epsilon}, \quad && on \quad \Omega \times \{t = 0\}, \\ & v(0,t) = v(1,t) = 0, \quad && on \quad \partial\Omega \times [0,T] \end{aligned} \tag{2}$$

2.1. Semidiscretization in Space

The semi discrete formulation involves approximation of the spatial variation of the dependent variable. The first step involves the construction of the weak form of the given problem over a typical element. In second step, we develop the finite element model by seeking approximation of the solution.

Set $V = H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. Applying Green's formula to problem (2) and using the boundary condition in the definition of V, we derive the variational form of problem (2). Find W: $(0,T] \to V$ such that

$$\int_a^b \left(k\frac{\partial W}{\partial x}\frac{\partial v}{\partial x} + W\frac{\partial v}{\partial t} - lWv\right)dx = 0 \tag{3}$$

Let us consider a uniform $1D$ mesh with the mesh size $h = x_i - x_{i-1}, \quad i = 1, \cdots, N$, which consists of $N+1$

points $a{=}x_0{<}x_1{<}\cdots{<}x_{N-1}{<}x_N{=}b$, we get

$$\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(k\frac{\partial W}{\partial x}\frac{\partial v}{\partial x}+W\frac{\partial v}{\partial t}-lWv)dx=0. \tag{4}$$

The weak formulation of equation (2) for an element (x_{i-1},x_i) is then given by,

$$\int_{x_{i-1}}^{x_i}(k\frac{\partial W}{\partial x}\frac{\partial v}{\partial x}+W\frac{\partial v}{\partial t}-lWv)dx=0 \tag{5}$$

Now, we define the finite dimensional subspace $V_h\subset V$, $V_h{=}span\{\psi_{i-1},\psi_i\}$ where

$$\psi_{i-1}(x)=\frac{x_i-x}{h}\qquad and\qquad \psi_i(x)=\frac{x-x_{i-1}}{h}$$

are the shape functions with $h{=}x_i-x_{i-1}$. Consider an approximate function $v(x)={=}a_0+a_1x$ so that each element has the variable defined as

$$v(x,t)=\begin{bmatrix}\psi_{i-1}(x) & \psi_i(x)\end{bmatrix}\begin{bmatrix}v_{i-1}(t)\\ v_i(t)\end{bmatrix} \tag{6}$$

Define the mapping $s{=}\frac{2x-x_1-x_2}{h}$. Then equation (6) becomes,

$$v(s,t)=\begin{bmatrix}\frac{1-s}{2} & \frac{1+s}{2}\end{bmatrix}\begin{bmatrix}v_{i-1}(t)\\ v_i(t)\end{bmatrix} \tag{7}$$

Substituting

$$W{=}\begin{bmatrix}\frac{1-s}{2}\\ \frac{1+s}{2}\end{bmatrix} \tag{8}$$

and equation (7) in equation (5) and Integrating we have,

$$\left\{\frac{k}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}-\frac{ah}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right\}\begin{bmatrix}v_{i-1}\\ v_i\end{bmatrix}+\frac{h}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\dot{v}_{i-1}\\ \dot{v}_i\end{bmatrix}=0 \tag{9}$$

The equation (9) is semidiscrete finite element equation for an element. Assembling contributions from all $N{=}5$ elements leads to the following global finite element equations:

$$[K]\{v\}+[M]\{\dot{v}\}{=}0 \tag{10}$$

where

$$[K]=\frac{k}{h}\begin{bmatrix}1 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & -1 & 2 & 1 & 0\\ 0 & 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & -1 & 1\end{bmatrix}-\frac{ah}{6}\begin{bmatrix}2 & 1 & 0 & 0 & 0 & 0\\ 1 & 4 & 1 & 0 & 0 & 0\\ 0 & 1 & 4 & 1 & 0 & 0\\ 0 & 0 & 1 & 4 & 1 & 0\\ 0 & 0 & 0 & 1 & 4 & 1\\ 0 & 0 & 0 & 0 & 1 & 2\end{bmatrix},$$

$$[M] = \frac{h}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad and \quad \{v\} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

2.2. Full Discretization in Time

Global semi discrete finite element equation is a system of differential equation in time.

$$[M]\{\dot{v}\} + [K]\{v\} = 0, \tag{11}$$

subject to the initial condition

$$v(x, 0) = 100x, 0 = x = 1 \tag{12}$$

where we fix $\epsilon = 1$. Full discretization of the global element equation is obtained by Crank-Nicolson method. As applied to a vector of time derivatives of the nodal values the weighted average of approximation on the equation(11),

$$[M]\left\{\frac{v_{n+1} - v_n}{\Delta t}\right\} + [K]\{v\} = 0 \tag{13}$$

Rearranging the equation (13),

$$[M]\{v_{n+1}\} = ([M] - \Delta t[K])\{v_n\}. \tag{14}$$

3. Numerical Results

The system of algebraic equation (14) is numerically solved in Matlab for the case $k = 1, l = 2, m = 1$.

$x(m)/t(units)$	$t = 0.2$	$t = 0.5$	$t = 1$
$x = 0$	350.37	875.94	1751.87
$x = 0.2$	-66.69	-196.72	-413.43
$x = 0.4$	87.24	158.12	276.20
$x = 0.6$	52.78	41.88	23.89
$x = 0.8$	206.59	396.72	712.98
$x = 1$	-210.02	-675.94	-1450.12

Values of $u(x, t)$

Figure 1. The behaviour of $u(x,t)$ with respect to different position for $\Delta t = 0.2$ units.

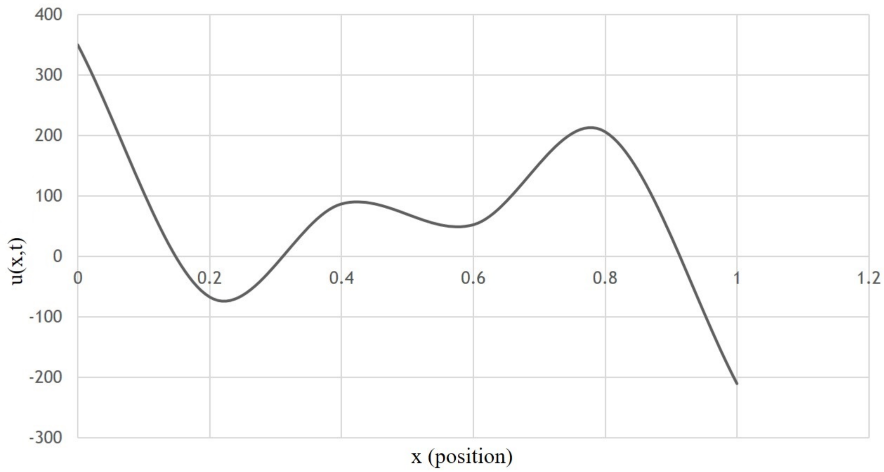


Figure 2. The behaviour of $u(x,t)$ with respect to different position for $\Delta t = 0.5$ units.

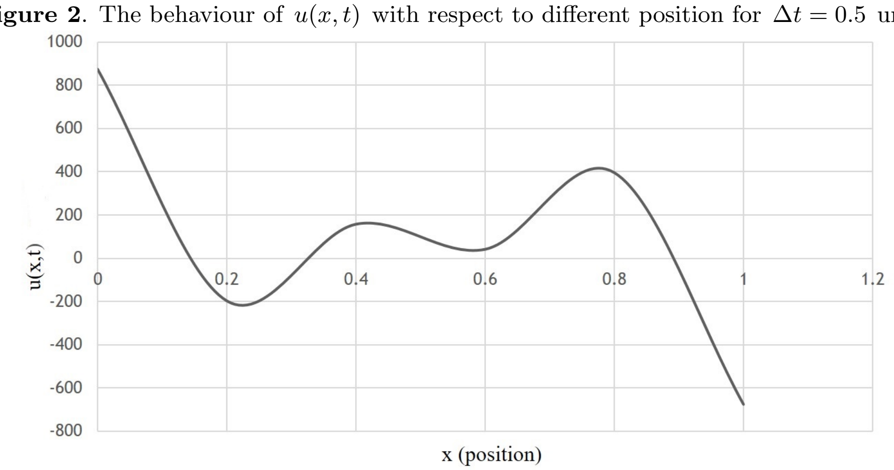


Figure 3. The behaviour of $u(x,t)$ with respect to different position for $\Delta t = 1.0$ units.

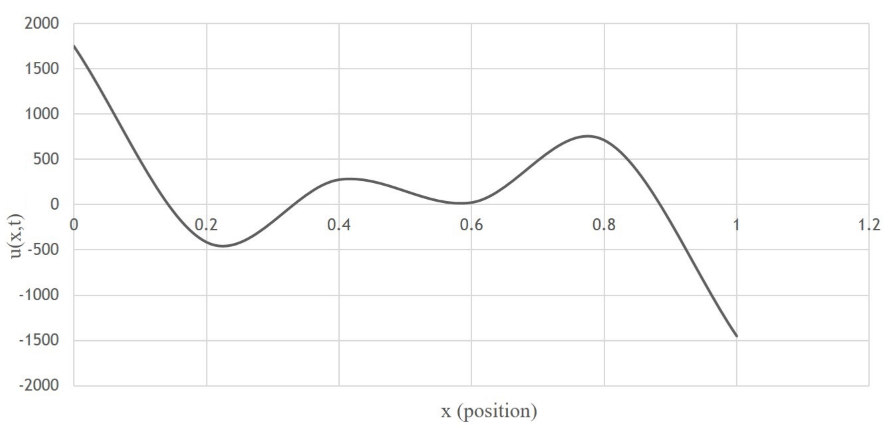
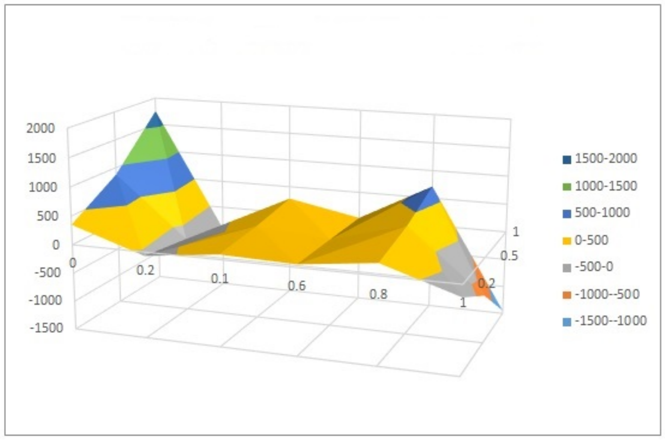


Figure 4. The behaviour of $u(x,t)$ with respect to different position for different time.



4. Conclusion

In this paper, we have found the numerical solution of Newell-Whitehead-Segel equation by Galerkin finite element method. Semidiscretization in space is done using Galerkin method and full discretization in time is made by Crank-Nicolson method. The numerical results are graphically interpreted.

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