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Endomorphism Rings Of H-supplemented Objects In Abelian Categories

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Abstract: Let $\mathcal{A}$ be an abelian category and M an object of $\mathcal{A}$. M is called a (D_3)-object if, for every direct summands A and B of M with $M = A + B$, $A \cap B$ is a direct summand of M. M is called a (D_4)-object if, $M = A \oplus B$ for subobjects A and B of M, and $f : A \to B$ is an epimorphism, then $\mathrm{Ker} f$ is a direct summand of A. We prove that every (D_3)-object is a (D_4)-object in an abelian category $\mathcal{A}$. Let $\mathcal{A}$ be an abelian category. Let $M = M_1 \oplus M_2$ be an object in $\mathcal{A}$. Assume that there is no nonzero direct summand of M_1 which is isomorphic to any one of M_2 and M is an H-supplemented (D_3)-object. Then $S/\nabla \cong S_1/\nabla_1 \times S_2/\nabla_2$, where $S = \mathrm{End}_{\mathcal{A}}(M), S_1 = \mathrm{End}_{\mathcal{A}}(M_1)$, $S_2 = \mathrm{End}_{\mathcal{A}}(M_2)$, $\nabla_1 = \{f \in S_1 : \mathrm{Im} f \ll M_1\}$ and $\nabla_2 = \{f \in S_2 : \mathrm{Im} f \ll M_2\}$. Moreover if M_1 is quasi-radical projective, then S_1/∇_1 is von Neumann regular; and if M_2 is d-square free, then S_2/∇_2 is reduced.

Key words: d-square free object, H-supplemented object, (D_3)-object, (D_4)-object, abelian category.

1. Introduction

Let $\mathcal{A}$ be an abelian category. Let M be an object of $\mathcal{A}$. A subobject A is said to be *small* in M, if $A+B \neq M$ for any proper subobject B of M and we write $A \ll M$ in this case. An object M is called H-*supplemented* if, for any subobject X of M, there exist a a direct summand A of M such that $M = X + Y$ if and only if $M = A + Y$, for all subobjects Y of M (see [11]). It is easy to prove that any object M in an abelian category $\mathcal{A}$ is H-supplemented if and only if for any subobject X of M, there exist a subobject X^* of M and a direct summand A of M such that $X^*/X \ll M/X$ and $X^*/A \ll M/A$. Now from [11] and [3] consider the conditions (D_3) and (D_4) for an object M:

(D_3) If A and B are direct summands of M such that $M = A + B$, then $A \cap B$ is a direct summand of M.

(D_4) If A and B are subobjects of M with $M = A \oplus B$ and $f : A \to B$ is an epimorphism, then $\mathrm{Ker} f$ is a direct summand of A.

An object M is said to be *d-square free* if whenever its factor object is isomorphic to $N^2 = N \oplus N$ for some object N, then $N = 0$.

Let $\mathcal{A}$ be an abelian category. An epimorphism $f : X \to Y$ is *small* if any morphism $g : X' \to X$ is an epimorphism provided that the composite fg is an epimorphism. This condition is equivalent to the following: If $U \subseteq X$ is a subobject with $U + \mathrm{Ker} f = X$, then $U = X$, namely $\mathrm{Ker} f \ll X$ (see [9]).

Let A and B be two objects in an abelian category $\mathcal{A}$. Then A is said to be *radical B-projective* if, for any morphism $f : A \to X$ and any epimorphism $g : B \to X$, there exists a morphism $h : A \to B$ such that

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$\mathrm{Im}(f-gh) \ll X$ (cf. [5], [6]). It is easy to see that A is radical B-projective if and only if for any epimorphism $g: B \to X$ and any morphism $f: A \to X$, there exist a small epimorphism $\rho: X \to Y$ for some object Y, a morphism $h: A \to B$ such that $\rho g h = \rho f$. The object A is said to be *radical projective* if A is radical M-projective for any object M, and A is said to be *quasi-radical projective* if A is radical A-projective.

Throughout this paper let S be the endomorphism ring of any object M in an abelian category $\mathcal{A}$. Note that $\nabla = \{f \in S : \mathrm{Im} f \ll M\}$ is a two sided ideal of S. Recall that a ring R is called *reduced* if it has no non-zero nilpotent elements and R is called *von Neumann regular* if for every element $x \in R$, there is an element $a \in R$ such that $xax = x$ (cf. [11]).

In [7], it is proven that

1. If M is a weakly supplemented d-square free module, then $\overline{S} = S/\nabla$ is reduced, and hence abelian ([7, Lemma 3.1]).
2. If M is an H-supplemented quasi-radical projective module, then S/∇ is von Neumann regular ([7, Lemma 3.2]).
3. If $M = A \oplus B$, where no nonzero direct summand of A is isomorphic to any one of B (for example, M is d-square free) and M is an H-supplemented (D_3)-module, then $f(A)$ is small in B for every homomorphism $f: A \to B$ ([7, Lemma 3.4]).
4. If $M = M_1 \oplus M_2$, there is no nonzero direct summand of M_1 which is isomorphic to any one of M_2 and M is an H-supplemented (D_3)-module, then $S/\nabla \cong S_1/\nabla_1 \times S_2/\nabla_2$, where $S = \mathrm{End}(M), S_1 = \mathrm{End}(M_1)$, $S_2 = \mathrm{End}(M_2)$, $\nabla_1 = \{f \in S_1 : \mathrm{Im} f \ll M_1\}$ and $\nabla_2 = \{f \in S_2 : \mathrm{Im} f \ll M_2\}$ ([7, Lemma 3.5]).

In this paper we will give the similar results to the above results in abelian categories (see Theorems 2.1, 2.2, Lemma 2.4 and Theorem 2.3). In particular, Lemmas 2.2 and 2.3 have different approaches in proving our results in abelian categories. We should note that, from the duality principle in abelian categories, the dual results to the results obtained in this paper are obtained easily. This is the motivation of our work, which allows one to deduce naturally properties of dual results on for example (C_3)-objects, (C_4)-objects and Goldie extending objects by the duality principle.

2. Results

First, we remember some backgrounds from [12]. Let $\mathcal{A}$ be an abelian category, $A' \subseteq A \in \mathcal{A}$. We write $\alpha(A')$ for the image of the composed morphism $A' \to A \xrightarrow{\alpha} B$. Assume that the square

$$\begin{array}{ccc} P & \xrightarrow{\gamma} & C \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback for f and g in $\mathcal{A}$. When f is a monomorphism, one may think of the pullback object P as the inverse image of A under g, and one often writes $g^{-1}(A)$ for P. In this language for the morphisms $A \xrightarrow{\beta} B \xrightarrow{\alpha} C$, $\mathrm{Ker}(\alpha\beta) = \beta^{-1}(\mathrm{Ker}\alpha)$. If both f and g are monomorphisms, then P is the greatest lower bound of A and C in $\mathcal{A}$, namely $P = A \cap C$. The following two results have the similar proofs with the proofs of [4, Lemma 3.4] and [7, Lemmas 3.1 and 3.2]. We give their proofs for sake of the completeness.

Theorem 2.1. (see [4, Lemma 3.4] and [7, Lemma 3.1]) *Let $\mathcal{A}$ be an abelian category and let $M \in \mathcal{A}$. If M is d-square free, then $\overline{S} = S/\nabla$ is reduced, and hence abelian, where $S = \mathrm{End}_{\mathcal{A}}(M)$ and $\nabla = \{f \in S : \mathrm{Im}f \ll M\}$.*

Proof. Let $\alpha \in S$ and $\alpha^2 \in \nabla$. Then $K = \mathrm{Im}\alpha^2 \ll M$. Assume that $M = \mathrm{Im}\alpha + L$ for any subobject L of M. Now $M/(\mathrm{Im}\alpha \cap L) = \mathrm{Im}\alpha/(\mathrm{Im}\alpha \cap L) \oplus L/(\mathrm{Im}\alpha \cap L)$ and $\mathrm{Im}\alpha = \mathrm{Im}\alpha^2 + \alpha(L) = K + \alpha(L)$. Note that

$$\mathrm{Im}\alpha/(\mathrm{Im}\alpha \cap L) \cong (\mathrm{Im}\alpha + L)/L = M/L \overset{\pi}{\to} M/(L+K)$$

where π is the induced epimorphism. On the other hand, we have the induced epimorphism $f : L/(\mathrm{Im}\alpha \cap L) \to M/(L+K)$. So there is an epimorphism from M to $[M/(L+K)]^2$. Since M is d-square free, $M = L + K$. Since $K \ll M$, $L = M$. This means that $\mathrm{Im}\alpha \ll M$. Thus $\alpha \in \nabla$. □

Theorem 2.2. (see [7, Lemma 3.2]) *Let $\mathcal{A}$ be an abelian category and let M be an H-supplemented quasi-radical projective object in $\mathcal{A}$. Then S/∇ is von Neumann regular, where $S = \mathrm{End}_{\mathcal{A}}(M)$ and $\nabla = \{f \in S : \mathrm{Im}f \ll M\}$.*

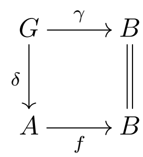
Proof. Let $f \in S$. Since M is H-supplemented and $\mathrm{Im}f$ is a subobject of M, there exist a decomposition $M = A \oplus B$ such that $M = \mathrm{Im}f + Y$ if and only if $M = A + Y$, for all subobjects Y of M. Let $p : M = A \oplus B \to A$ be the canonical projection. By $M = A \oplus B$, $M = \mathrm{Im}f + B$ and hence $pf : M \to A$ is an epimorphism. Again as M is H-supplemented and $\mathrm{Ker}(pf)$ is a subobject of M, there exist a decomposition $M = K \oplus T$ such that $M = \mathrm{Ker}(pf) + Y$ if and only if $M = K + Y$, for all subobjects Y of M. Then $M = \mathrm{Ker}(pf) + T$ implies that $(pf)|_T : T \to A$ is a small epimorphism. Since A is radical T-projective, there exist a morphism $g : A \to T$ and a small epimorphism $\rho : A \to Y$ for some object Y such that $\rho 1_A = \rho(pf)|_T g$. Then $\mathrm{Im}(f - fgpf) = f(\mathrm{Ker}(pf)) + (f - fgpf)(T)$ because $M = \mathrm{Ker}(pf) + T$. Since $f(\mathrm{Ker}(pf)) \subseteq B \cap \mathrm{Im}f \ll B$, it suffices to show $(f - fgpf)(T) \ll M$. By $\rho 1_A = \rho(pf)|_T g$, $(\rho p(f - fgpf))(T) = 0$ and hence $(1 - gpf)(T) \subseteq \mathrm{Ker}(\rho pf)$. By $\mathrm{Ker}(\rho(pf)|_T) \ll T$, $(f - fgpf)(T) = f(1 - gpf)(T) \subseteq f(\mathrm{Ker}(\rho pf)) \ll f(T) \subseteq M$. □

Lemma 2.1. *Let $\mathcal{A}$ be an abelian category, $M \in \mathcal{A}$ and A a direct summand of M. Then A is also H-supplemented if M is an H-supplemented (D_3)-object.*

Proof. Assume that $M = A \oplus A'$ and M is an H-supplemented (D_3)-object. We want to see that A is H-supplemented. Let Y be a subobject of A. Consider the subobject $Y \oplus A'$ of M. Since M is H-supplemented, there exist a direct summand X' of M such that $M = (Y \oplus A') + L$ if and only if $M = X' + L$, for all subobjects L of M. Note that $M = A \oplus A' = (Y \oplus A') + A = X' + A$. Since M is (D_3), $X' \cap A$ is a direct summand of M, and hence it is a direct summand of A. Now we will show that $A = Y + K$ if and only if $A = (X' \cap A) + K$, for all subobjects K of A. Let K be a subobject of A and $A = Y + K$. Now $M = A \oplus A' = (Y + K) + A' = (Y \oplus A') + K = X' + K$. The subobject lattice of any object in an abelian category is modular (see [12, Proposition 5.3]). So we have that $A \cap M = A = A \cap (X' + K) = K + (A \cap X')$. Now let $A = (X' \cap A) + K$. Then $M = (X' \cap A) + K + A' = X' + (K \oplus A') = (Y \oplus A') + (K \oplus A') = (Y + K) \oplus A'$. Again from the modularity, $A = A \cap ((Y + K) \oplus A') = (Y + K) + (A \cap A') = Y + K$. □

Lemma 2.2. *Let $\mathcal{A}$ be an abelian category, $M \in \mathcal{A}$. If M is a (D_3)-object, then it is a (D_4)-object.*

Proof. Let $M = A \oplus B$ and $f : A \to B$ be an epimorphism. Consider the following pullback square

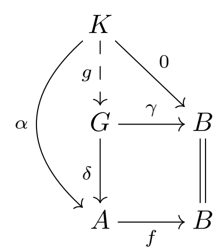


in $\mathcal{A}$. Then from [1, Proposition 2.12], the sequence

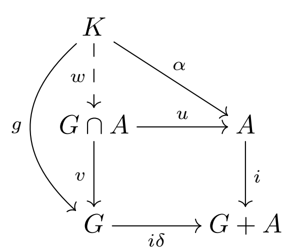
$$0 \longrightarrow G \xrightarrow{\left[\begin{smallmatrix}\delta\\ \gamma\end{smallmatrix}\right]} A \oplus B \xrightarrow{[\,f\ -1\,]} B \longrightarrow 0$$

is exact in $\mathcal{A}$ ($[\,f\ -1\,]$ is an epimorphism and from pullback property, $\left[\begin{smallmatrix}\delta\\ \gamma\end{smallmatrix}\right]$ is a kernel of $[\,f\ -1\,]$).

Let $K = \mathrm{Ker} f$ and $\alpha = ker f$. Now we will show that $K = G \cap A$. From pullback property, we get the following commutative diagram with the unique morphism $g : K \to G$ satisfying $\gamma g = 0$ and $\delta g = \alpha$:

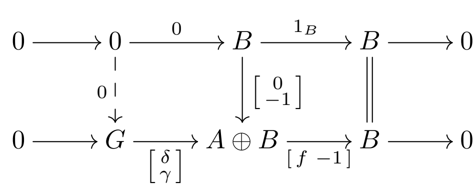


Note that δ is a monomorphism. Then since $G \cap A$ is a pullback of $i\delta$ and $i : A \to G + A$, where i is the inclusion morphism again from pullback property, we get the following commutative diagram with the unique morphism $w : K \to G \cap A$ satisfying $uw = \alpha$ and $vw = g$:



Since α is a monomorphism, w is a monomorphism. By the universal property of the intersection, $K = G \cap A$.

One may construct the following commutative diagram



where the rows are short exact sequences. By [10, Proposition 13.2], the left square of the above diagram is a pullback, and hence again from [1, Proposition 2.12], the sequence

$$0 \longrightarrow 0 \longrightarrow G \oplus B \longrightarrow A \oplus B \longrightarrow 0$$

is exact. Therefore $G \oplus B \cong A \oplus B$, namely G is a direct summand of M.

Now one may construct the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K = G \cap A & \xrightarrow{\alpha} & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & {\scriptstyle v=g}\downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \| & & \\ 0 & \longrightarrow & G & \xrightarrow[\left[\begin{smallmatrix}\delta\\\gamma\end{smallmatrix}\right]]{} & A \oplus B & \xrightarrow[{[\,f \ -1\,]}]{} & B & \longrightarrow & 0 \end{array}$$

where the rows are exact sequences. By [1, Proposition 2.12], the left square of the above diagram is a pushout. But the following square

$$\begin{array}{ccc} G \cap A & \xrightarrow{\alpha} & A \\ {\scriptstyle g}\downarrow & & \downarrow \\ G & \longrightarrow & G + A \end{array}$$

is also a pushout of α and g. Hence $M = A \oplus B = G + A$. Now since M is (D_3), $\mathrm{Ker} f = G \cap A$ is a direct summand of M. Thus M is a (D_4)-object in $\mathcal{A}$. □

Lemma 2.3. *Let $\mathcal{A}$ be an abelian category, $M \in \mathcal{A}$ and $M = K \oplus L$. If M is a (D_4)-object, then K is a (D_4)-object in $\mathcal{A}$.*

Proof. Assume that $K = A \oplus B$ and $f : A \to B$ is an epimorphism. Let $\alpha = ker f$. One may construct the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker} f \oplus L & \xrightarrow{\left[\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right]} & A \oplus L & \xrightarrow{[\,f \ 0\,]} & B & \longrightarrow & 0 \\ & & {\scriptstyle [\,1 \ 0\,]}\downarrow & & \downarrow{\scriptstyle [\,1 \ 0\,]} & & \| & & \\ 0 & \longrightarrow & \mathrm{Ker} f & \xrightarrow[\alpha]{} & A & \xrightarrow[f]{} & B & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$. By [1, Proposition 2.12], the left square of the above diagram is a pushout. Since M is (D_4), the upper short exact sequence splits. Since pushouts preserve split sequences, the lower short exact sequence splits. Hence K is a (D_4)-object. □

Lemma 2.4. (see [4, Proposition 2.18] and [7, Lemma 3.4]) *Let $\mathcal{A}$ be an abelian category. Let $M = A \oplus B$ be an object in $\mathcal{A}$, where no nonzero direct summand of A is isomorphic to any one of B (for example, M is d-square free). Let M be an H-supplemented (D_3)-object. Then* $\mathrm{Im} f$ *is small in B for every morphism $f : A \to B$.*

Proof. Let $f : A \to B$ be a morphism such that $\mathrm{Im} f$ is not small in B. Since B is H-supplemented by Lemma 2.1, there exist a decomposition $B = B_1 \oplus B_2$ such that $B = B_1 + Y$ if and only if $B = \mathrm{Im} f + Y$, for all subobjects Y of B. Let $p : B = B_1 \oplus B_2 \to B_1$ be the canonical projection. Then $pf : A \to B_1$ is an epimorphism. Note that $M = (A \oplus B_1) \oplus B_2$. By Lemmas 2.2 and 2.3, $A \oplus B_1$ is (D_4). Thus there exists a subobject $A_1 \subseteq A$ such that $A = A_1 \oplus \mathrm{Ker}(pf)$. Hence $A_1 \cong A/\mathrm{Ker}(pf) \cong B_1$. Since there is no nonzero direct summand of A which is isomorphic to any one of B, $B_1 = 0$ and hence $\mathrm{Im} f \ll B$.

Note that for d-square free case we don't need the "H-supplemented (D_3)-object" conditions by [4, Proposition 2.18]. □

Theorem 2.3. *Let $\mathcal{A}$ be an abelian category. Let $M = M_1 \oplus M_2$ be an object in $\mathcal{A}$. Assume that there is no nonzero direct summand of M_1 which is isomorphic to any one of M_2 and M is an H-supplemented (D_3)-object. Then $S/\nabla \cong S_1/\nabla_1 \times S_2/\nabla_2$, where $S = \mathrm{End}_{\mathcal{A}}(M), S_1 = \mathrm{End}_{\mathcal{A}}(M_1)$, $S_2 = \mathrm{End}_{\mathcal{A}}(M_2)$, $\nabla_1 = \{f \in S_1 : \mathrm{Im} f \ll M_1\}$ and $\nabla_2 = \{f \in S_2 : \mathrm{Im} f \ll M_2\}$. Moreover if M_1 is quasi-radical projective, then S_1/∇_1 is von Neumann regular; and if M_2 is d-square free, then S_2/∇_2 is reduced.*

Proof. By Lemma 2.4, $S/\nabla \cong S_1/\nabla_1 \times S_2/\nabla_2$. By Theorem 2.1, S_2/∇_2 is reduced and by Theorem 2.2 and Lemma 2.1, S_1/∇_1 is von Neumann regular. □

We finish this paper with the following two examples showing that the results in this paper work for some objects in abelian categories which are not the module categories.

Example 2.1. Let $\mathcal{C}$ be a finitely accessible additive category. Then there is a Grothendieck category $\mathcal{A}(\mathcal{C})$ (uniquely determined up to equivalence). This category $\mathcal{A}(\mathcal{C})$ is equivalent to the category $(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ of all contravariant additive functors from the full subcategory $\mathrm{fp}(\mathcal{C})$ of finitely presented objects of $\mathcal{C}$ to the category Ab of abelian groups (see [2, Theorem 1.4]). This Grothendieck category $\mathcal{A}(\mathcal{C})$ is an abelian category which is not the module category.

Example 2.2. Let $\mathcal{C}$ be an exactly definable additive category. Then there is a locally coherent Grothendieck category $\mathcal{D}(\mathcal{C})$ (uniquely determined up to equivalence). This category $\mathcal{D}(\mathcal{C})$ is equivalent to the category $\mathrm{Lex}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$ of left exact contravariant additive functors from $\mathcal{A}$ to the category Ab of abelian groups for some skeletally small abelian category $\mathcal{A}$ (see [8, Lemma 1.1]). This Grothendieck category $\mathcal{D}(\mathcal{C})$ is an abelian category which is not the module category.

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